Partial Differential Equations - Assignment 1

Rafael Pacheco
77128580

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1 Problem 1*

Description:

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0, \qquad x^{(0)} = \sqrt[3]{20}, \qquad Iterations = I = 4 \tag{1.1}$$

Newton's method:

$$\begin{aligned} x^{(k+1)} &= x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} \\ f'(x) &= 3x^2 + 4x + 10 \\ x^{(k+1)} &= x^{(k)} - \frac{x^3 + 2x^2 + 10x - 20}{3x^2 + 4x + 10} \end{aligned} \tag{1.2}$$

In addition the relative error and roots are calculated by:

$$\begin{aligned} f(x) = x^3 + 2x^2 + 10x - 20 &= (x - 1.3688)(x^2 - 14.6113) = 0 \\ x_r &= \begin{bmatrix} -1.6844 + 3.4313i \\ -1.6844 - 3.4313i \\ 1.3688 + 0.0000i \end{bmatrix} \\ e_r^{(k)} &= \frac{x^{(k+1)} - x_r}{x_r} \end{aligned} \tag{1.3}$$

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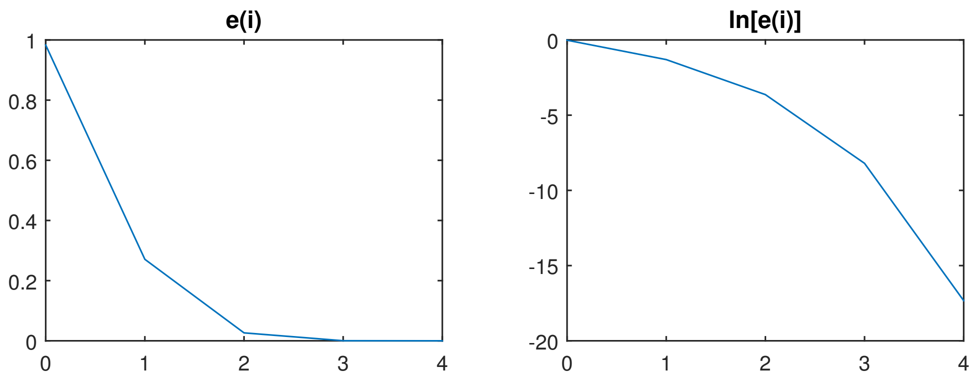
Note that the root of interest is the third one which is real. Using Matlab to iterate from $x^{(0)}$ to $x^{(4)}$, same for the relative error, the results are:

$$\begin{bmatrix} k & x^{(k)} & e_r^{(k)} \\ 0 & 2.7144 & 0.9831 \\ 1 & 1.7396 & 0.2709 \\ 2 & 1.4050 & 0.0264 \\ 3 & 1.3692 & 0.0003 \\ 4 & 1.3688 & 0.0000 \end{bmatrix} \tag{1.4}$$

Then plotting the relative error in linear and logarithmic scale:



2 Problem 5*

Third-order numerical quadratures in intervals (0,1).

a) Minimum number of integration points, and specify the integration points and weights.

Gauss quadrature: $2n+2$ dof $\xrightarrow{order}$ $2n+1=3, \quad n=1$
The number of points is $n+1=2$

Therefore the integration points (z_i) and weights (w_i) are:

$$z_i = (-1^i)\sqrt{\frac{1}{3}}, \qquad w_i = 1, \qquad i = 1,2 \quad | \quad i \in \mathbb{Z} \tag{2.1}$$

B) Is it possible to obtain a third-order quadrature with the following four integration points: $x_0 = 1/4, x_1 = 1/2, x_2 = 3/4$ and $x_3 = 1$? If it is possible, compute the corresponding weights; otherwise, justify why not.

A priori, since the interval is from 0 to 1, thus the domain is semi-open. But since the points are equally spaced and re-arranging the domain for a third-order from [1/4, 1] , the expression obtained is no other than Simpson's second rule:

$$I = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

$$h = \frac{(x_3 - x_0)}{2n+1} = \frac{1-\frac{1}{4}}{2+1} = \frac{1}{4} \tag{2.2}$$

$$\begin{bmatrix} i & 0 & 1 & 2 & 3 \\ w_i & \frac{3}{8} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} \end{bmatrix}$$

3 Problem 6*

A) If n+1 points Gaussian quadrature is used for numerical integration state the order of the polynomial that is integrated exactly

Number of points : $n+1$
Order: $2n+1$

B) n=2, which of the following integrals will be integrated exactly?

$$n = 2 \xrightarrow{order} 2*2+1 = 5$$

i) $\int_o^1 sin(x)dx$ → No, since $sin(x) \approx \sum_{(2n+1,i=1)}^{i=3} \frac{x^i}{i!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$, $7 > 5$

ii) $\int_o^1 x^3 dx$ → Yes, $4 < 5$

iii) $\int_o^1 x^3 dx$ → Yes, $3 < 5$

iv) $\int_o^1 x^{5.5} dx$ → No, $5.5 > 5$

4 Problem 7*

$$\int_0^1 12x dx, \qquad \int_0^1 (5x^3+2x)d \tag{4.1}$$

intervals = 2

Trapezoidal rule

$$\begin{gathered}
m=2, \qquad h=\frac{b-a}{m}=\frac{1-0}{2}=\frac{1}{2} \\
I_i=\frac{h}{2}\Big[f(x_{i-1})+f(x_i)\Big] \\
I=\frac{h}{2}\Big[f(x_0)+2\big(\sum_{i=1}^{m-1} f(x_i)\big)+f(x_m)\Big] \\
I_I=\frac{1}{4}\Big[0+2\big(6\big)+12\Big]=6, \qquad E=\alpha f''(x)\rightarrow\frac{d^2(12x)}{dx^2}=0 \\
I_{II}=\frac{1}{4}\Big[0+2\big(5(\frac{1}{2})^3+1\big)+7\Big]=\frac{41}{16}, \qquad E=-\frac{(b-a)^3}{12m^2}f''(\mu)=-\frac{5}{8}\mu=-\frac{5}{16}
\end{gathered} \tag{4.2}$$

Simpson's rule

$$\begin{gathered}
m=2, \qquad h=\frac{b-a}{2m}=\frac{1-0}{2*2}=\frac{1}{4} \\
I_i=\frac{h}{3}\Big[f(x_{2i-2})+4f(x_{2i-1})+f(x_{2i})\Big] \\
I=\frac{h}{3}\sum_{i=1}^{m}\Big[f(x_{2i-2})+4f(x_{2i-1})+f(x_{2i})\Big] \\
I_I=\frac{1}{12}\Big[[0+4(3)+6]+[6+4(9)+12]\Big]=6, \qquad E=\alpha f^{4)}(x)\rightarrow\frac{d^4(12x)}{dx^4}=0 \\
I_{II}=\frac{1}{12}\Big[[0+20(\frac{1}{4})^3+\frac{8}{4}+5(\frac{1}{2})^3+\frac{2}{2}]+[5(\frac{1}{2})^3+\frac{2}{2}+20(\frac{3}{4})^3+\frac{24}{4}+5(1)^3+2(1)\Big]=\frac{9}{4} \\
E=\alpha f^{4)}(x)\rightarrow\frac{d^4(5x^3+2x)}{dx^4}=0
\end{gathered} \tag{4.3}$$

The methods behave as expected.

5 Problem 10*

Perform the numerical integration of

$$\int_0^1 \int_0^1 (9x^3 + 8x^2)(y^3 + y)dxdy \tag{5.1}$$

using Simpson's rule in each direction.Is the approximation behaving as expected?

To integrate this equation, first the value of the function on $x = [0,1]$ and $y = [0,1]$ will be obtained. Thus Simpson's rule for $m = 1, n = 1$ will be performed in order to avoid the error term:

$$I = \frac{h^2}{9mn}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90}f^{4)}(\mu), \quad f^{4)}(x,y) = 0$$

$$x_i = \frac{i}{2}, \quad y_j = \frac{j}{2}, \quad i = 0,...,2, \quad j = 0,...,2; \quad i,j \in \mathbb{Z}$$

$$f(x_i, y_j) = \begin{cases} f(0, y_j) = [0,0,0] \\ f(1, y_j) = [0, 6.25, 34] \\ f(x_i, 0) = [0,0,0,0] \\ f(x_i, 1) = [0, 10.625, 34] \end{cases} \tag{5.2}$$

$$h = \frac{1}{2}$$

$$I = \frac{1}{36}[0 + 0 + 4(\frac{25}{4}) + 34 + 0 + 0 + 4(\frac{85}{8}) + 34] = \frac{271}{72}$$

Therefore there is no error and the approximation is the same as the analytical answer.