Partial differential equations — List 3 — Rafael Pacheco 77128580N

2*.
$$\begin{cases} u_t + a u_x = 0, & x \in (0,1),\ t \geq 0,\ a > 0 \\ u(x,0) = \sin(2\pi x) \\ u(0,t) = u(1,t) \end{cases}$$

a) Scheme
- Implicit → Backward.
- Δt → Backward difference in time.
- Δx → Forward difference in space

• Scheme: BTFS

• BT: $$u_t \Big|_i^{n+1} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)$$

• FS: $$u_x \Big|_i^{n+1} = \frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + O(\Delta x)$$

• BTFS:
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + O(\Delta t) + O(\Delta x) = 0$$

- Truncating error neglected:
- $r = \frac{\Delta t}{\Delta x}$
- $(1 + ar)\, u_i^{n+1} - ar\, u_{i-1}^{n+1} = u_i^n \quad , \quad i = 1, \dots, I+1$

b) periodic Boundary ?

$$u(0,t) = u_0^n = (1+ar)\, u_0^{n+1} - ar\, u_{-1}^{n+1}$$

- $u_{-1}^n = u_I^n$

Therefore :

$$u_0^n = (1+ar)\, u_0^{n+1} - ar\, u_I^{n+1}$$

- The system to solve is :

$$\underline{\underline{A}} \cdot \underline{u}^{n+1} = \underline{\underline{I}}\, \underline{u}^n$$

$$[\underline{\underline{A}}] = \begin{bmatrix} 1+ar & 0 & \cdots & \cdots & -ar \\ -ar & 1+ar & 0 & \cdots & 0 \\ 0 & -ar & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & -ar & 1+ar \end{bmatrix}$$

- row $I+1 \equiv 0$, so it can be neglected !

c) Direct method $\rightarrow$ $\underline{\underline{A}} = \underline{\underline{L}} \cdot \underline{\underline{U}}$ (Doolittle)

$\Downarrow$

- No symmetry
- But diagonal dominant

$\underline{\underline{L}} \equiv$ Lower triangular matrix

$\underline{\underline{U}}$ Upper " "

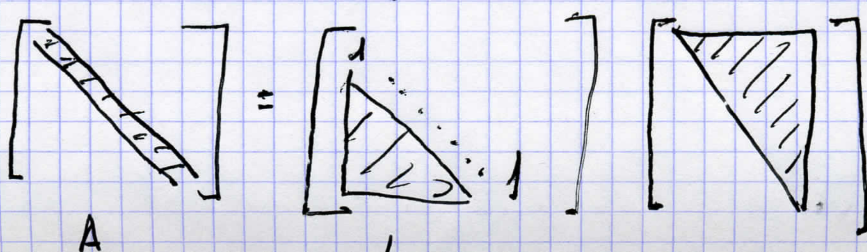
Iterative $\rightarrow$ (Gauss - Seidel)

$\Downarrow$

- No symmetry
- Faster than Jacobi method

d) Fill in for direct method

- ones in the diagonal



4*
$$\begin{cases} u_t = \nu u_{xx} + \sigma u & \text{in } x \in (0,1),\ t>0 \\ & \sigma<0,\ \nu>0 \\ u(0,t)=0 \quad , \quad u_x(1,t)=0 & ;\ \text{B.C.} \\ u(x,0) = \begin{cases} 0 & x < 1/4 \\ 4x-1 & 1/4 \le x < 1/2 \\ -4x+3 & 1/2 \le x < 3/4 \\ 0 & 3/4 \le x \end{cases} \end{cases}$$

a) explicit FD $\rightarrow$ FTCS $\rightarrow$ $O(\Delta t, \Delta x^2)$

FT: $$u_t\Big|_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)$$

CS: $$u_{xx}\Big|_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + O(\Delta x^2)$$

- Truncation error neglected $[O(\Delta t, \Delta x^2)]$

FTCS: $$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\nu}{\Delta x^2} L(u_i^n) + \sigma u_i^n$$

- $\frac{\Delta t}{\Delta x^2} = r$

$$u_i^{n+1} = \nu r L(u_i^n) + \sigma \Delta t\, u_i^n + u_i^n$$

$$u_i^{n+1} = \nu r L(u_i^n) + u_i^n(\sigma \Delta t + 1)$$

therefore applying the B.C. and I.C.:

- I.C. $\rightarrow$ $u_i^0 = \cancel{u(x,0)}\ u(\Delta x \cdot i, 0)$; $i = 0, \dots, I+1$
- Dirichlet $\rightarrow$ $u_0^{n+1} = 0$, $n \ge 0$
- Neumann $\rightarrow$ $u_{I+1}^{n+1} = \nu r L(u_{I+1}^n) + u_{I+1}^n(\sigma \Delta t + 1)$

- Developing $L(u_{I+1}^n)$!

$$u_{I+1}^{n+1} = \nu r\Big(u_I^n - 2u_{I+1}^n + \underbrace{u_{I+2}^n}_{\text{fictitional}}\Big) + u_{I+1}^n(\sigma \Delta t + 1)$$

Using a CS:

$$u_x\Big|_{I+1}^p = \frac{u_{I+2}^p - u_I^p}{2\Delta x} = 0 \rightarrow u_{I+2}^p = u_I^p$$

Therefore substituting in the general equation with Neumann Condition approximated by Central diff.

$$u_{I+1}^{n+1} = 2\nu r\, u_I^n + (1+\sigma\Delta t - 2\nu r)\, u_{I+1}^n$$

Finally, the numerical scheme is:

- $u_i^{n+1} = \nu r\, L(u_i^n) + (1+\sigma\Delta t)\, u_i^n, \quad n>0,\ i=1,\dots,I$
- $u_{I+1}^{n+1} = 2\nu r\, u_I^n + (1+\sigma\Delta t - 2\nu r)\, u_{I+1}^n, \quad n>0$
- $u_0^{n+1} = 0, \quad n>0$
- $u_i^0 = u(\Delta x \cdot i, 0), \quad i=0,\dots, I+1$

b) $\sigma = 0$ (Diffusion eqn)

- Initial & Dirichlet remain the same:
  - $u_i^{n+1} = \nu r L(u_i^n) + u_i^n, \quad n>0,\ i=1,\dots,I$
  - $u_{I+1}^{n+1} = 2\nu r\, U_I^n + (1-2\nu r)\, u_{I+1}^n, \quad n>0$

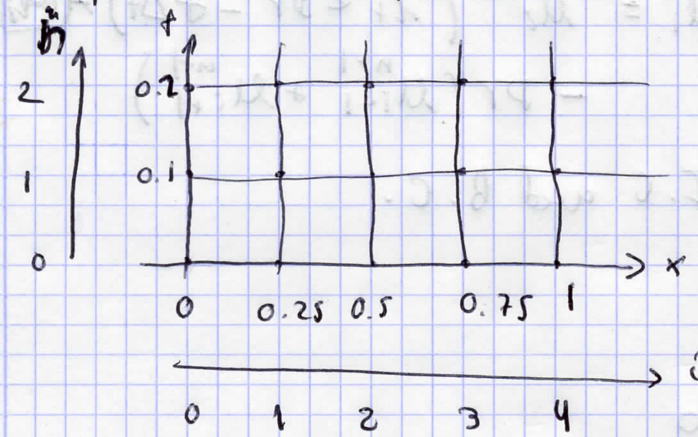
$\nu = 0$ (Reaction eqn)

- Initial & Dirichlet remain the same:
  - $u_i^{n+1} = (1+\sigma)\, u_i^n, \quad n>0,\ i=1,\dots,I$
  - $u_{I+1}^{n+1} = (1+\sigma\Delta t)\, u_{I+1}^n, \quad n>0$

c) $\nu = 0.1$, $\sigma = -0.1$, $\Delta x = 0.25$, $\Delta t = 0.1$

- 2 steps

The mesh obtained is:

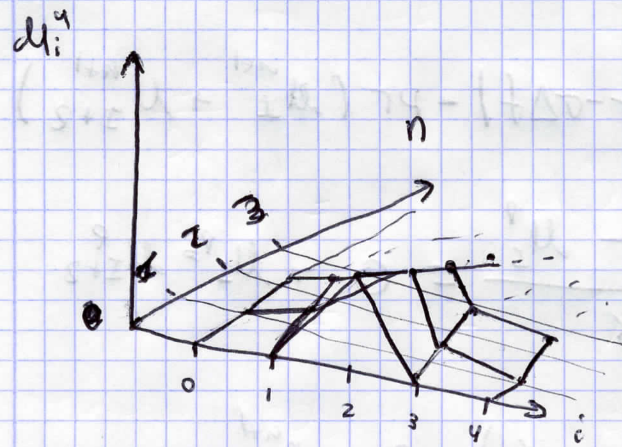


$$\begin{cases} \to u_0^n = 0, \quad n \geq 0 \\ \to u_i^0 = u(x_i, 0), \quad i = 0,1,2,3,4 \\ \to u_i^{n+1} = \nu r\, u_{i-1}^n + (1+\sigma\Delta t - 2\nu r)\, u_i^n + \nu r\, u_{i+1}^n, \quad i = 1,2,3, \; n \geq 0 \\ \quad \begin{cases} \nu r = \frac{4}{25} \\ (1+\sigma\Delta t - 2\nu r) = \frac{67}{100} \end{cases} \\ \to u_4^{n+1} = \frac{8}{25} u_3^n + \frac{67}{100} u_4^n \end{cases}$$

i →

u_i^n	0	1	2	3	4
0	0	0	1	0	0
1	0	$\frac{4}{25}$	$\frac{67}{100}$	$\frac{4}{25}$	0
2	0	$\frac{134}{625}$	0.5001	$\frac{134}{625}$	$\frac{32}{625}$

n ↓



- There is a diffusion effect starting on node $i=3$ and passing its effect to the neighbours ($\nu > 0$)
- Also the effect is a loss since $\sigma < 0$.

d) BTCS:

$$BT: \; u_t\big|_i^{n+1} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)$$

$$CS: \; u_{xx}\big|_i^{n+1} = \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2} + O(\Delta x^2)$$

$$\cancel{\boxed{\begin{aligned} &BTCS: \; \left(r = \frac{\Delta t}{\Delta x^2}\right) \\ &u_i^{n+1} - u_i^n = \nu r \left(u_{i+1}^{n+1} + u_{i-1}^{n+1}\right) (2r + \sigma \Delta t)\, u_i^n \\ &u_i^{n+1} - \nu r \left(u_{i+1}^{n+1} + u_{i-1}^{n+1}\right) = (1 - 2r + \sigma \Delta t)\, u_i^n \end{aligned}}}$$

BTCS:

$$u_i^{n+1} - u_i^n = \nu r \left(u_{i-1}^{n+1} + u_{i+1}^{n+1} - 2u_i^{n+1} \right) + \sigma \Delta t u_i^{n+1}$$

$$\cancel{u_i^n = \frac{1}{1+\sigma\Delta t}} \qquad u_i^n = u_i^{n+1}\left(1+2\nu r - \sigma\Delta t\right) \cancel{\Delta t u_i}$$
$$- \nu r \left(u_{i-1}^{n+1} + u_{i+1}^{n+1} \right)$$

- Applaying again I.C and B.C.

Dirichlet

$u_0^{n+1} = 0 \quad , n \geq 0$

IC

$u_i^0 = u(\Delta x i, 0) \quad , \quad i = 0, \dots, I+1$

Neumann

1) compute $I+1$:

$$u_{I+1}^n = u_{I+1}^{n+1}\left(1+2\nu r - \sigma\Delta t\right) - \nu r\left(u_I + u_{I+2} \right)$$

2) CD:

$$u_x\Big|_{I+1}^p = \frac{u_{I+2}^p - u_I^p}{2\Delta x} = 0 \to u_I^p = u_{I+2}^p$$

3) Back to "1)"

$$u_{I+1}^n = u_{I+1}^{n+1}\left(1+2\nu r - \sigma\Delta t\right) - 2\nu r\, u_I^{n+1}$$

therefore:

$$\begin{cases} u_i^{n} = u_i^{n+1}\left(1+2\nu r - \sigma\Delta t\right) - \nu r\left(u_{i-1}^{n+1} + u_{i+1}^{n+1} \right), & n \geq 0, \; i = 1, \dots, I \\ u_{I+1}^n = u_{I+1}^{n+1}\left(1+2\nu r - \sigma\Delta t\right) - 2\nu r\, u_I^{n+1}, & n \geq 0 \\ u_i^0 = u(\Delta x i, 0), & i = 0, \dots, I+1 \\ u_0^n = 0, & n \geq 0 \end{cases}$$

The structure of the matrix will be:

$$\underline{\underline{A}}\;\underline{u}^{n+1} = \underline{\underline{I}}\;\underline{u}^{n}$$

$$[\underline{\underline{A}}] = \begin{bmatrix} 1+2\nu r-\sigma\Delta t & -\nu r & 0 & 0 \\ -\nu r & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \\ 0 & & -2\nu r & \end{bmatrix}$$

$$\underset{\text{N.B.C.}}{\downarrow}$$

- Most suitable method to solve?
  - Tridiagonal $\longrightarrow$ Thomas algorithm (n operations)