2016. 12. 4. ANQI LI.

Numerical Methods for PDEs. — ODE.

Q1.

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \qquad ; \ \theta(1) = 0.4\,\text{rad}, \ \frac{d\theta}{dt}(1) = 0\ \text{rad/s}$$

$L = 1\,m, \quad g = 9.8\,m/s^2$

(a).

2^{nd} order Runge-Kutta.

$$y^*_{i+1} = y_i - h*f(y_i, t_i)$$

$$y_{i+1} = y_i - \frac{h}{2}\left[f(y_i, t_i) + f(y^*_{i+1}, t_{i+1})\right]$$

2 time step. - $h = 0.5$.

Initial values. $\theta(1) = 0.4$ rad, $\frac{d\theta}{dt}(1) = 0$ rad/s.

$$\bar{\theta} = \begin{bmatrix} \theta \\ \frac{d\theta}{dt} \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} ; \quad \bar{f}(\theta, t) = \begin{bmatrix} d\theta_1/dt \\ d\theta_2/dt \end{bmatrix} = \begin{bmatrix} \frac{d\theta}{dt} \\ \frac{d^2\theta}{dt^2} \end{bmatrix} = \begin{bmatrix} \theta_2 \\ -\frac{g}{L}\theta_1 \end{bmatrix}$$

1^{st} step. $t = 0.5s$.

$$\bar{\theta}^*_1 = \bar{\theta}_0 - h \cdot \bar{f}(\theta_0, t_0) = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}_0 - 0.5 \times \begin{bmatrix} 0 \\ -\frac{9.8}{1} \times 0.4 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 1.96 \end{bmatrix}$$

$$\theta_1 = \bar{\theta}_0 - \frac{h}{2} \times \left[f(\theta_0, t_0) + f(\theta^*_1, t_1)\right]$$

$$= \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}_0 - \frac{0.5}{2} \times \left(\begin{bmatrix} 0 \\ -\frac{9.8}{1} \times 0.4 \end{bmatrix} + \begin{bmatrix} 1.96 \\ -\frac{9.8}{1} \times 0.4 \end{bmatrix} \right) = \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix}$$

2^{st} step. $t = 0s$.

$$\theta^*_2 = \bar{\theta}_1 - h \times \bar{f}(\theta_1, t_1) = \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix}_1 - 0.5 \times \begin{bmatrix} 1.96 \\ -\frac{9.8}{1} \times (-0.09) \end{bmatrix} = \begin{bmatrix} -1.07 \\ 1.519 \end{bmatrix}$$

$$\theta_2 = \theta_1 - \frac{h}{2} \times \left[f(\theta_1, t_1) + f(\theta^*_1, t_1)\right]$$

$$= \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix} - \frac{0.5}{2} \times \left(\begin{bmatrix} 1.96 \\ -\frac{9.8}{1} \times (-0.09) \end{bmatrix} + \begin{bmatrix} 1.519 \\ -\frac{9.8}{1} \times (-1.07) \end{bmatrix} \right) = \begin{bmatrix} -0.9598 \\ -0.8820 \end{bmatrix}$$

4 time step . $-h=0.25$

Initial values. $\bar{\theta}_0 = \begin{bmatrix} \theta(0) \\ \frac{d\theta}{dt}(0) \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$

1st step. $t=0.75s$

$$\bar{\theta}_1^* = \bar{\theta}_0 - h\cdot\bar{f}(\theta_0, t_0) = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - 0.25\times\begin{bmatrix} 0 \\ -\frac{9.8}{1}\times 0.4 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.98 \end{bmatrix}$$

$$\bar{\theta}_1 = \bar{\theta}_0 - \frac{h}{2}\cdot\left[\bar{f}(\theta_0, t_0) + \bar{f}(\theta_1^*, t_1)\right]$$

$$= \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - \frac{0.25}{2}\times\left(\begin{bmatrix} 0 \\ -\frac{9.8}{1}\times 0.4 \end{bmatrix} + \begin{bmatrix} 0.98 \\ -\frac{9.8}{1}\times 0.4 \end{bmatrix}\right) = \begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix}$$

2nd Step. $t=0.5s$.

$$\bar{\theta}_2^* = \bar{\theta}_1 - h\cdot\bar{f}(\theta_1, t_1) = \begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix} - 0.25\times\begin{bmatrix} 0.98 \\ -\frac{9.8}{1}\times 0.2775 \end{bmatrix} = \begin{bmatrix} 0.0325 \\ 1.66 \end{bmatrix}$$

$$\bar{\theta}_2 = \bar{\theta}_1 - \frac{h}{2}\cdot\left(f(\theta_2^*, t_2) + f(\theta_1, t_1)\right)$$

$$= \begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix} - \frac{0.25}{2}\times\left(\begin{bmatrix} 0.98 \\ -\frac{9.8}{1}\times 0.2775 \end{bmatrix} + \begin{bmatrix} 1.66 \\ -\frac{9.8}{1}\times 0.0325 \end{bmatrix}\right) = \begin{bmatrix} -0.0525 \\ 1.3598 \end{bmatrix}$$

3rd step. $t=0.25s$.

$$\bar{\theta}_3^* = \bar{\theta}_2 - h\cdot\bar{f}(\theta_2, t_2) = \begin{bmatrix} -0.0525 \\ 1.3598 \end{bmatrix} - 0.25\times\begin{bmatrix} 1.3598 \\ -\frac{9.8}{1}\times(-0.0525) \end{bmatrix} = \begin{bmatrix} -0.3924 \\ 1.2312 \end{bmatrix}$$

$$\bar{\theta}_3 = \bar{\theta}_2 - \frac{h}{2}\cdot\left[f(\theta_2, t_2) + f(\theta_3^*, t_3)\right]$$

$$= \begin{bmatrix} -0.0525 \\ 1.3598 \end{bmatrix} - \frac{0.25}{2}\cdot\left(\begin{bmatrix} 1.3598 \\ -\frac{9.8}{2}\times(-0.0525) \end{bmatrix} + \begin{bmatrix} 1.2312 \\ -\frac{9.8}{2}\times(-0.3924) \end{bmatrix}\right) = \begin{bmatrix} -0.3763 \\ 0.8147 \end{bmatrix}$$

4th step. $t=0s$.

$$\bar{\theta}_4^* = \bar{\theta}_3 - h\cdot\bar{f}(\theta_3, t_3) = \begin{bmatrix} -0.3763 \\ 0.8147 \end{bmatrix} - 0.25\times\begin{bmatrix} 0.8147 \\ -\frac{9.8}{1}\times 0.3763 \end{bmatrix} = \begin{bmatrix} -0.58 \\ -0.1073 \end{bmatrix}$$

$$\bar{\theta}_4 = \bar{\theta}_3 - \frac{h}{2}\left[f(\theta_3, t_3) + f(\theta_4^*, t_4)\right]$$

$$= \begin{bmatrix} -0.3763 \\ 0.8147 \end{bmatrix} - \frac{0.25}{2}\left(\begin{bmatrix} 0.8147 \\ -\frac{9.8}{1}\times 0.3763 \end{bmatrix} + \begin{bmatrix} -0.1073 \\ -\frac{9.8}{1}\times(-0.58) \end{bmatrix}\right) = \begin{bmatrix} -0.4648 \\ -0.3568 \end{bmatrix}$$

(b). Relative Error.

At $t=0s$. $\theta_{4step} = -0.4648$. $\theta_{2step} = -0.9598$

$$E_{relative} = \left|\frac{\theta_{2step} - \theta_{4step}}{\theta_{4step}}\right| = \left|\frac{-0.9598-(-0.4648)}{-0.4648}\right| = 1.065$$

(c).

$$\frac{\overset{\swarrow tol}{E_{h^*}}}{E_h} = \frac{C(h^*)^{p+1}}{Ch^{p+1}} \longrightarrow h^* = \left(\frac{tol}{E_h}\right)^{\frac{1}{p+1}} \cdot h.$$

$h=0.25$. For RK2, the worder of the method $p=2$.

In order to obtain a relative error three orders of magnitude smaller.

$$h^* = (10^{-3})^{\frac{1}{2+1}} \times 0.25 = 0.025.$$

Q2.

$$\frac{dy}{dx} = y - x^2 + 1 \ , \ x \in (0,1) \ ; \ y(0) = 1$$

(a). Forward Euler. $h=0.25$

$$y_{i+1} = y_i + h \cdot f(y_i, x_i)$$

1st step. — $x_0 = 0$

$$y_1 = y_0 + h \cdot f(y_0, x_0) = 1 + 0.25 \times (1 - 0^2 + 1) = 1.5$$

2nd step. — $x_1 = 0.25$.

$$y_2 = y_1 + h \cdot f(y_1, x_1) = 1.5 + 0.25 \times (1.5 - 0.25^2 + 1) = 2.11$$

3rd step. — $x_2 = 0.5$

$$y_3 = y_2 + h f(y_2, x_2) = 2.11 + 0.25 \times (2.11 - 0.5^2 + 1) = 2.825.$$

4th step — $x_3 = 0.75$

$$y_4 = y_3 + h f(y_3, x_3) = 2.825 + 0.25 \times (2.825 - 0.75^2 + 1) = 3.64$$

(b). Heun method.

Same computational cost, so the h for Heum method is 0.5.

~~h=0.5,~~

1^{st} time step.

$$y_1^* = y_0 + hf(y_0, x_0) = 1 + (1-0^2+1) \times 0.5 = 2.$$

$$y_1 = y_0 + \frac{h}{2}\left(f(y_0, x_0) + f(y_1^*, x_1)\right) = 1 + \frac{0.5}{2}\left[(1-0^2+1) + (2-0.5^2+1)\right]$$

$$= 2.1875.$$

2^{nd} time step.

$$y_2^* = y_1 + hf(y_1, x_1) = 2.1875 + 0.5 \times (2.1875^2 - 0.5^2 + 1) = 3.656$$

$$y_2 = y_1 + \frac{h}{2}\left(f(y_1, x_1) + f(y_2^*, x_2)\right)$$

$$= 2.1875 + \frac{0.5}{2} \cdot \left[(2.1875 - 0.5^2 + 1) + (3.656 - 1^2 + 1)\right] = 3.836$$

(c). –

From (b). three points are obtained. (0,1). (0.5, 2.1875). (1, 3.836). the pure enterpolation of the polynomial ~~with~~ will be quadratic

$$y = 0.9219x^2 + 1.9141x + 1.$$

(d). For ~~the first~~ (a) 5 points are obtained, which can also be fitted by pure enterpolation with order of 4. ~~or~~.

However, the best fitting method will be usiy be fit to find a quadratic function.

Q3.

$$\frac{dy}{dx} = f(x, y)$$

Forward Euler. $Y_{i+1} = Y_i + h f(x_i, Y_i)$.

(a). Truncation Error.

$$Y_{i+1} = Y_i + h \frac{dY}{dx}(x_i) + O(h^2)$$

$$\frac{dY}{dx}(x_i) = \frac{Y_{i+1} - Y_i}{h} + O(h).$$

The truncation error of the scheme is $O(h)$.

The method is said to be consistence.

$$\tau_i(h) = \frac{1}{h} R_i(h).$$

when $h \to 0$, $\max_{0 \le i \le m} \tau_i(h) \to 0$.

$\therefore$ the method is consistent.

(b). Backward Euler

$$Y_{i+1} = Y_i + h f(x_{i+1}, Y_{i+1})$$

(c) Stability

$$\frac{dy}{dx} = -\lambda y$$

$$y_{i+1} = y_i - h \cdot \lambda y_i = (1 - h\lambda) y_i \quad ①$$

Add the perturbation on both side.

$$y_{i+1} + P_{i+1} = (1 + h\lambda)(y_i + P_i) \quad ②$$

Obtained from ① & ② $\Rightarrow$ $P_{i+1} = (1 + h\lambda) P_i$

For stable. $|P_{i+1}| \le |P_i|$

$$\Rightarrow |1 - h\lambda| \le 1. \quad -1 \le 1 - h\lambda \le 1$$

$$\Rightarrow \boxed{0 \le h\lambda \le 2}$$

(d). Backward Euler.

$$\frac{dy}{dx} = -25y^{3.5} \quad ; y(0)=1, \quad h = 1/10.$$

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1}) = y_i + h\cdot(-25y_{i+1}^{3.5}).$$

Newton-Raphson Method. ($u = y_{i+1}$)

$$g(u) = u - y_i - h\cdot f(x_{i+1}, y_{i+1})$$

$$\delta u = -\frac{g(u)}{g'(u)} = -\frac{u - y_i - hf(x_{i+1},}{1 - h\cdot\frac{\partial f(x_{i+1}, u)}{\partial u}}$$

$$\delta u = -\frac{u - y_i + h\times 25u^{3.5}}{1 + h\times 25\times 3.5\times u^{2.5}}$$

$$u_{i+1} = u_i + \delta u_i$$

1st time step.

$h=0.1$. $i=0$. $y(0)=1$. $u_0 = y_0 = 1$.

Iter 1.

$$\delta u_0 = -\frac{u_0 - y_0 + h\times 25u_0^{3.5}}{1 + h\times 87.5u_0^{2.5}} = -0.2564$$

$$u_1 = u_0 + \delta u_0 = 1 - 0.2564 = 0.7436.$$

Iter 2.

$$\delta u_1 = -\frac{u_1 - y_0 + h\times 25u_1^{3.5}}{1 + h\times 87.5u_1^{2.5}} = -0.1218$$

$$u_2 = u_1 + \delta u_1 = 0.7436 - 0.1218 = 0.6218$$

$\therefore y_1 = 0.6218.$

2nd time step.

$h = 0.1$. $i=0$. $y_1 = 0.6218$. $u_0 = y_1 = 0.6218$.

Iter 1.

$$\delta u_0 = -\frac{u_0 - y_1 + h\times 25u_0^{3.5}}{1 + h\times 87.5u_0^{2.5}} = -0.1292.$$

$$u_1 = u_0 + \delta u_0 = 0.6218 - 0.1292 = 0.4926.$$

Iter 2.

$$\delta u_1 = \frac{u_1 - y_1 + h\times 25u_1^{3.5}}{1 + h\times 87.5u_1^{2.5}} = -0.0323.$$

$$u_2 = u_1 + \delta u_1 = 0.4926 - 0.0323 = 0.4603.$$

$\therefore y_2 = 0.4603.$

(e). Forward Euler. $h=0.1$. $y_0=1$.

1^{st} time step.

$$y_1 = y_0 + h\times(-25y_0^{3.5}) = 1+0.1\times(-25\times1^{3.5}) = -1.5.$$

2^{nd} time step.

$$y_2 = y_1 + h\times(-25y_1^{3.5}) = -1.5+0.1\times[-25\times(-1.5)^{3.5}]$$

$$= -1.5+10.333\,i. \longrightarrow \text{complex number.}$$

(f).

The maximum stable interval, according to matlab is $h = 1/30$.

Q4.

$$\frac{d^2y}{dx^2}+\omega^2 y = 0. \qquad y(0)=0\ ;\ \frac{dy}{dx}(0)=\omega.$$

(a). System of first order ODEs.

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ \frac{dy}{dx} \end{bmatrix},\quad f(y,x) = \begin{bmatrix} \frac{dy}{dx} \\ \frac{d^2y}{dx^2} \end{bmatrix} = \begin{bmatrix} y_2 \\ -\omega^2 y_1 \end{bmatrix}$$

(b). Set. $\omega^2=3$.

Forward Euler. 4 steps. $h=0.25$.

Initial Conditions.

$$\bar{y}_0 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_0 = \begin{bmatrix} y(0) \\ \frac{dy}{dx}(0) \end{bmatrix}_0 = \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}$$

1^{st} time step.

$$\bar{y}_1 = \bar{y}_0 + hf(y_0,x_0) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_0 + h\times\begin{bmatrix} y_2 \\ -\omega^2 y_1 \end{bmatrix}_0 = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix} + 0.25\times\begin{bmatrix} \sqrt{3} \\ -3\times0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/4 \\ \sqrt{3} \end{bmatrix}$$

2^{nd} time step.

$$\bar{y}_2 = \bar{y}_1 + hf(x_1,y_1) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_1 + h\times\begin{bmatrix} y_2 \\ -\omega^2 y_1 \end{bmatrix}_1 = \begin{bmatrix} \sqrt{3}/4 \\ \sqrt{3} \end{bmatrix} + 0.25\times\begin{bmatrix} \sqrt{3} \\ -3\times\frac{\sqrt{3}}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \sqrt{3}-\frac{3\sqrt{3}}{16} \end{bmatrix}$$

3rd time step

$$\bar{y}_3 = \bar{y}_2 + h \times \bar{f}(x_2, y_2) = \begin{bmatrix} \sqrt{3}/2 \\ \sqrt{3} - \frac{3\sqrt{3}}{16} \end{bmatrix} + 0.25 \begin{bmatrix} \sqrt{3} - \frac{3\sqrt{3}}{16} \\ -3 \times \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 1.2178 \\ 0.7578 \end{bmatrix}$$

4th time step.

$$\bar{y}_4 = \bar{y}_3 + h \times \bar{f}(x_3, y_3) = \begin{bmatrix} 1.2178 \\ 0.7578 \end{bmatrix} + 0.25 \times \begin{bmatrix} 0.7578 \\ -3 \times 1.2178 \end{bmatrix} = \begin{bmatrix} 1.4073 \\ -0.1556 \end{bmatrix}$$

(c).

$$\frac{E_{h^*}}{E_h} = \frac{C(h^*)^{p+1}}{C h^{p+1}} \longrightarrow h^* = \left(\frac{tol}{E_h}\right)^{\frac{1}{p+1}} \cdot h.$$

with 8 steps. $h = \frac{1}{8}$.

For Forward Euler, the order of the method is $p=1$.

∴ New time step.

$$h^* = (10^{-3})^{\frac{1}{1+1}} \times \frac{1}{8} = \frac{1}{8} \times 10^{-\frac{3}{2}}$$

$$= 0.00395.$$