2017.1.1 ANQI LI.

Numerical Methods for Partial Differential Equations.

Exercise - Finite Differences.

Q2. Differential equation. $u_t + a u_x = 0$, $x \in (0,1)$. $t \geq 0$. $a > 0$.

Initial Condition. $u(x,0) = \sin(2\pi x)$.

Boundary Condition. $u(0,t) = u(1,t)$

(a) Implicit finite difference scheme. first order in time and space.

Backward in time. $\left.\frac{\partial u}{\partial t}\right|_{i+1}^{n+1} = \frac{u_{i+1}^{n+1} - u_{i+1}^{n}}{\Delta t} + O(\Delta t)$.

Backward in space. $\left.\frac{\partial u}{\partial x}\right|_{i+1}^{n+1} = \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} + O(\Delta x)$.

$$\Rightarrow \frac{u_{i+1}^{n+1} - u_{i+1}^{n}}{\Delta t} + a \cdot \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} = 0.$$

$$\left(1 + a\frac{\Delta t}{\Delta x}\right) u_{i+1}^{n+1} - a\frac{\Delta t}{\Delta x} u_i^{n+1} = u_{i+1}^{n}$$

(b). System of equation to ~~solove~~ solve.

$$\begin{bmatrix} -a\frac{\Delta t}{\Delta x} & \left(1+a\frac{\Delta t}{\Delta x}\right) & 0 & \cdots & 0 \\ 0 & -a\frac{\Delta t}{\Delta x} & \left(1+a\frac{\Delta t}{\Delta x}\right) & \cdots & 0 \\ 0 & 0 & -a\frac{\Delta t}{\Delta x} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \left(1+a\frac{\Delta t}{\Delta x}\right) \end{bmatrix} \cdot \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_M^{n+1} \end{bmatrix} = \begin{bmatrix} u_2^{n} \\ u_3^{n} \\ u_4^{n} \\ \vdots \\ u_{M+1}^{n} \end{bmatrix}$$

Periodic boundary condition.

$u_1^{n+1} = u_M^{n+1}$,

$\therefore$ remove the last equation.

$$\begin{bmatrix} -a\frac{\Delta t}{\Delta x} & \left(1+a\frac{\Delta t}{\Delta x}\right) & 0 & \cdots & 0 \\ 0 & -a\frac{\Delta t}{\Delta x} & \left(1+a\frac{\Delta t}{\Delta x}\right) & \cdots & 0 \\ 0 & 0 & -a\frac{\Delta t}{\Delta x} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -a\frac{\Delta t}{\Delta x} \end{bmatrix} \cdot \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_{M-1}^{n+1} \end{bmatrix} = \begin{bmatrix} u_2^{n} \\ u_3^{n} \\ u_4^{n} \\ \vdots \\ u_M^{n} \end{bmatrix}$$

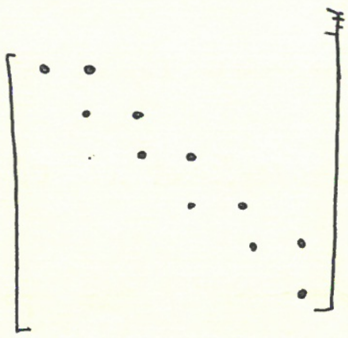
(c). Direct method and iterative method for the solution of linear systems of equation.

Direct method. — Backward Substitution.

Iterative method. —— Gauss-Seidel method.

(d). Draw schematically the fill-in of the matrix

$$\begin{bmatrix} -a\frac{\Delta t}{\Delta x} & (1+a\frac{\Delta t}{\Delta x}) & 0 & \cdots & 0 \\ 0 & -a\frac{\Delta t}{\Delta x} & (1+a\frac{\Delta t}{\Delta x}) & \cdots & 0 \\ 0 & 0 & -a\frac{\Delta t}{\Delta x} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & -a\frac{\Delta t}{\Delta x} \end{bmatrix} \Rightarrow$$



Q4. Diffusion-reaction PDE.

$$u_t = \omega u_{xx} + \sigma u \qquad \text{in } x \in (0,1),\ t>0.$$

Boundary Condition. $u(0,t) = 0$. and $u_x(1,t) = 0$

Initial Condition. $u(x,0) = \begin{cases} 0 & x < 1/4 \\ 4x-1 & 1/4 \le x < 1/2 \\ -4x+3 & 1/2 \le x < 3/4 \\ 0 & 3/4 \le x. \end{cases}$

(a). Propose an explicit finite difference scheme.

FTCS: $\frac{\partial u}{\partial t}\Big|_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t}$

$$\frac{\partial^2 u}{\partial x^2}\Big|_i^n = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2}$$

$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} = \omega \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} + \sigma u_i^n$$

$$u_i^{n+1} = \omega r u_{i-1}^n + (1 - 2\omega r + \Delta t \sigma) u_i^n + r \omega u_{i+1}^n \quad ; r = \frac{\Delta t}{\Delta x^2}$$

~~For~~ Boundary Condition ~~Eqx~~:

$$u(0,t)=0.$$

Initial Condition. $u(x,0) = \begin{cases} 0 & x<1/4 \\ 4x-1 & 1/4 \le x < 1/2 \\ -4x+3 & 1/2 \le x < 3/4 \\ 0 & 3/4 \le x. \end{cases}$

Assume M nodes, when $i=M$.

$$u_M^{n+1} = r\omega u_{M-1}^n + (1-2r\omega+\sigma\Delta t)u_M^n + r\omega u_{M+1}^n$$

Use boundary condition $u_x(1,t)=0$.

$$u_x(1,t) = \frac{\partial u}{\partial x}\Big|_{i=M}^{t} = \frac{u_{M+1}^n - u_{M-1}^n}{2\Delta x} = 0. \qquad \boxed{u_{M+1}^n = u_{M-1}^n}$$

$$\Rightarrow u_M^{n+1} = 2r\omega u_{M-1}^n + (1-2r\omega+\sigma\Delta t)u_M^n$$

(b) Which scheme is obtained for $\sigma=0$ & $\omega=0$.

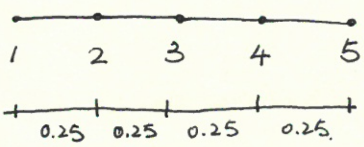
For $\sigma=0$, equation become. $u_t=\omega u_{xx}$. (diffusion equation)

FTCS. Forward in time, centred in space.

For $\omega=0$, equation become. $u_t=\sigma u$. (reaction equation).

Equation became an ODE. so can use forward ~~/back~~ Euler to solve the problem.

(c) Compute 2 time step with $\omega=0.1$. $\sigma=-0.1$. $\Delta x=0.25$. $\Delta t=0.1$



1st time step. $t=\Delta t=0.1$.

$\omega=0.1$. $\sigma=-0.1$ $\Delta x=0.25$, $\Delta t=0.1$.

$$r=\frac{\Delta t}{\Delta x^2} = \frac{0.1}{0.25^2} = 1.6$$

Initial values. $u_0^0=0$ $u_2^0=0$ $u_3^0=1$ $u_4^0=0$. $u_5^0=0$.

~~$u_1^1 = \omega r u_0^0 +$~~ Boundary Condition. $u_1^i=0$.

$$u_2^1 = \omega r u_1^0 + (1-2\omega r+\Delta t\sigma)u_2^0 + r\omega u_3^0 = 1.6\times0.1\times1 = 0.16.$$

$$u_3^1 = \omega r u_2^0 + (1-2\omega r+\Delta t\sigma)u_3^0 + r\omega u_4^0 = [1-2\times0.1\times0.16+0.1\times(-0.1)]\times1 = 0.958$$

$u_4^1 = \Delta r u_3^0 + (1-2\Delta r+\Delta t\sigma) u_4^0 + r\Delta u_5^0 \quad = 0.1\times 1.6\times 1 = 0.16$

$u_5^1 = 2\Delta r u_4^0 + (1-2\Delta r+\Delta t\sigma) u_5^0 = 0.$

2nd time step.

~~Ur Initial Condi~~ Boundary Condition. $u_1^2 = 0.$

$u_2^2 = \Delta r u_1^1 + (1-2\Delta r+\Delta t\sigma) u_2^1 + r\Delta u_3^1 = 0.958\times 0.16 + 0.16\times 0.958 = 0.31$

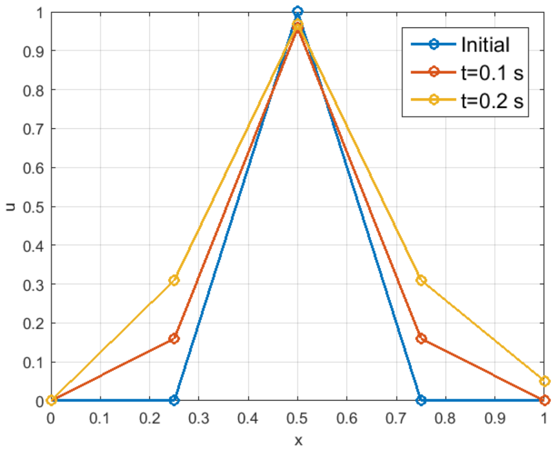
$u_3^2 = \Delta r u_2^1 + (1-2\Delta r+\Delta t\sigma) u_3^1 + r\Delta u_4^1 = 0.16^2 + 0.958^2 + 0.16^2 = 0.969$

$u_4^2 = \Delta r u_3^1 + (1-2\Delta r+\Delta t\sigma) u_4^1 + r\Delta u_5^1 = 0.958\times 0.16 + 0.16\times 0.958 = 0.31$

$u_5^2 = 2\Delta r u_4^1 + (1-2\Delta r+\Delta t\sigma) u_5^1 = 2\times 0.16^2 = 0.0512.$

~~(d). Propose an implicit finite difference scheme.~~

Graphic of the profile of u



The result obtained is reasonable. Initially, the pollutant is in the centre and during the time it start to diffuse to the side. The left boundary is fixed as 0, so the pollutant spread more on the right boundary which is not fixed.

(d) Propose an implicit finite difference scheme.

BTCS: $\left.\frac{\partial u}{\partial t}\right|_i^{n+1} = \frac{u_i^{n+1} - u_i^n}{\Delta t}$

$\left.\frac{\partial^2 u}{\partial t^2}\right|_i^{n+1} = \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2}$

$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} = \mu \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2} + \sigma u_i^{n+1}$

~~$(\mu \frac{\Delta t}{\Delta x^2}) u_{i-1}^{n+1} - (1 + 2\mu \frac{\Delta t}{\Delta x^2}) u_i^{n+1} + \mu \frac{\Delta t}{\Delta x^2}$~~

$(-\mu \frac{\Delta t}{\Delta x^2}) u_{i-1}^{n+1} + (1 + 2\mu \frac{\Delta t}{\Delta x^2} - \sigma \Delta t) u_i^{n+1} - \mu \frac{\Delta t}{\Delta x^2} u_{i+1}^{n+1} = u_i^n$

$\Rightarrow -\mu r u_{i-1}^{n+1} + (1 + 2\mu r - \sigma \Delta t) u_i^{n+1} - \mu r u_{i+1}^{n+1} = u_i^n \quad ; \quad r = \frac{\Delta t}{\Delta x^2}$

Boundary Condition.

$u(0,t) = 0 \quad , \quad u_1^n = u_1^{n+1} = 0.$

Initial Condition.

$$u(x,0) = \begin{cases} 0 & x < 1/4 \\ 4x - 1 & 1/4 \le x \le 1/2 \\ -4x + 3 & 1/2 \le x < 3/4 \\ 0 & 3/4 \le x \end{cases}$$

Similar as ~~ex~~ FTCS, assume M nodes, when $i = M$

$-\mu r u_{M-1}^{n+1} + (1 + 2\mu r - \sigma \Delta t) u_M^{n+1} - \mu r u_{M+1}^{n+1} = u_M^n$

Use Boundary Condition $u_x(1,t) = 0$

$u_x(1,t) = \left.\frac{\partial u}{\partial x}\right|_{i=M}^{t} = \frac{u_{M+1}^{n+1} - u_{M-1}^{n+1}}{2\Delta x} = 0 \quad \Rightarrow \boxed{u_{M+1}^{n+1} = u_{M-1}^{n+1}}$

$\Rightarrow -2\mu r u_{M-1}^{n+1} + (1 + 2\mu r - \sigma \Delta t) u_M^{n+1} = u_M^n$

Matrix formulation

$$\begin{bmatrix} 1+2\Delta r-\sigma\Delta t & -\Delta r & 0 & \cdots & 0 & 0 \\ -\Delta r & 1+2\Delta r-\sigma\Delta t & -\Delta r & \cdots & 0 & 0 \\ 0 & -\Delta r & 1+2\Delta r-\sigma\Delta t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2\Delta r-\sigma\Delta t & -\Delta r \\ 0 & 0 & 0 & \cdots & -2\Delta r & 1+2\Delta r-\sigma\Delta t \end{bmatrix} \times \begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ \vdots \\ u_{M-1}^{n+1} \\ u_M^{n+1} \end{bmatrix} = \begin{bmatrix} u_2^{n} \\ u_3^{n} \\ u_4^{n} \\ \vdots \\ u_{M-1}^{n} \\ u_M^{n} \end{bmatrix}$$

The linear system of equation can be solved by using Gauss~~ian~~ method.