

Numerical Methods for Partial Differential Equations

ODEs

Exercise 2

Computational Mechanics 2016-2017

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1. The motion of a non-frictional pendulum is governed by the Ordinary Differential Equation (ODE)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where θ is the angular displacement, $L = 1$ m is the pendulum length and the gravity acceleration is $g = 9.8$ m/s². The position and velocity at time $t = 1$ s are known:

$$\frac{d\theta}{dt}(1) = 0 \frac{\text{rad}}{\text{s}} \quad \theta(1) = 0.4 \text{ rad}$$

- a) Solve the initial boundary value problem in the interval $(0, 1)$ using a second-order Runge-Kutta method to determine the initial position at $t = 0$ s, with 2 and 4 time steps.

In the first step we should transform the 2nd order ODE to a system of 1st order ODEs to be able to use 2nd order Runge-Kutta (Heun) method.

$$y_1 = y \rightarrow y_1' = y' = y_2$$

$$y_2' = y'' = -9.8 y = -9.8 y_1$$

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -9.8 y_1 \end{bmatrix}$$

Now having this system of ODEs we may use the Heun method as:

$$Y_{i+1} = Y_i + \frac{h}{2} [k_1 + k_2]$$

$$k_1 = f(x_i, Y_i)$$

$$k_2 = f(x_i + h, Y_i + h k_1)$$

So for the first try we will use 2 steps with hand implementing $h = -0.5$ (the minus sign is because we are going backward from the end of domain to the first of it)

$$Y_{1_1} = 0.4$$

$$Y_{1_2} = Y_{1_1} + \frac{h}{2} [k_{1_1} + k_{1_2}]$$

- $k_{1_1} = Y_{2_1} = 0$
- $k_{1_2} = Y_{2_1} + h \cdot k_{2_1}$
- $k_{2_1} = -9.8 * y_{1_1} = -3.92$

- $k_{1_2} = Y_{2_1} + h.k_{2_1} = 0 + (-0.5) * (-3.92) = 1.96$
- $Y_{1_2} = Y_{1_1} + \frac{h}{2}[k_{1_1} + k_{1_2}] = 0.4 + \frac{-0.5}{2}[0 + 1.96] = -0.09$

$$Y_{2_2} = Y_{2_1} + \frac{h}{2}[k_{2_1} + k_{2_2}]$$

- $k_{2_1} = -9.8 * y_{1_1} = -3.92$
- $k_{2_2} = -9.8 * (y_{1_1} + h.k_{1_1}) = -9.8(0.4 - 0.5 * 0) = -3.92$
- $Y_{2_2} = Y_{2_1} + \frac{h}{2}[k_{2_1} + k_{2_2}] = 0 + \frac{-0.5}{2}[-3.92 - 3.92] = 1.96$

$$Y_{1_3} = Y_{1_2} + \frac{h}{2}[k_{1_1} + k_{1_2}]$$

- $k_{1_1} = Y_{2_2} = 1.96$
- $k_{1_2} = Y_{2_2} + h.k_{2_1}$
- $k_{2_1} = -9.8 * y_{1_2} = -9.8 * (-0.09) = 0.882$
- $k_{1_2} = Y_{2_2} + h.k_{2_1} = 1.96 + (-0.5) * (0.882) = 1.519$
- $Y_{1_3} = Y_{1_2} + \frac{h}{2}[k_{1_1} + k_{1_2}] = -0.09 + \frac{-0.5}{2}[1.96 + 1.519] = -0.9598$

Using coded program for Euler method we would have this table which completely proves calculations above.

h=-0.5 [2 steps]										
xn	y.1	k1	y.2	k2	y1n	y.1	k1	y.2	k2	y2n
1					0.4000					0.0000
0.5	0.000	0.000	1.960	1.960	-0.0900	0.400	-3.920	0.400	-3.920	1.9600
0	1.960	1.960	1.519	1.519	-0.9598	-0.090	0.882	-1.070	10.486	-0.8820

So for the second try we will use 4 steps implementing h=-0.25. The summary is provided in the following table:

h=-0.25 [4 steps]										
xn	y.1	k1	y.2	k2	y1n	y.1	k1	y.2	k2	y2n
1					0.4000					0.0000
0.75	0.000	0.000	0.980	0.980	0.2775	0.400	-3.920	0.400	-3.920	0.9800
0.5	0.980	0.980	1.660	1.660	-0.0525	0.278	-2.720	0.033	-0.319	1.3598
0.25	1.360	1.360	1.231	1.231	-0.3763	-0.052	0.514	-0.392	3.846	0.8147
0	0.815	0.815	-0.107	-0.107	-0.4648	-0.376	3.688	-0.580	5.684	-0.3568

- b) Using the approximations obtained in a), compute an approximation of the relative error in the solution computed with 2 steps.

By solving the 2nd order ODE regarding the B.C.s we the analytical solution would be:

$$\theta = C_2 \sin(\sqrt{9.8}x) + C_1 \cos(\sqrt{9.8}x)$$

$$\frac{d\theta}{dt} = \sqrt{9.8}C_2 \cos(\sqrt{9.8}x) - \sqrt{9.8}C_1 \sin(\sqrt{9.8}x)$$

$$\frac{d\theta}{dt}(1) = 0 \frac{\text{rad}}{\text{s}} \quad \theta(1) = 0.4 \text{ rad}$$

$$C_1 = 0.4 \cos \sqrt{9.8}$$

$$C_2 = 0.4 \sin \sqrt{9.8}$$

$$\theta = 0.4 \sin \sqrt{9.8} \sin(\sqrt{9.8}x) + 0.4 \cos \sqrt{9.8} \cos(\sqrt{9.8}x)$$

$$\theta(0) = 0.4 \sin \sqrt{9.8} \sin(0) + 0.4 \cos \sqrt{9.8} \cos(0) \cong -\mathbf{0.4} \quad \text{Exact Solution}$$

Approximation of the relative error :

$$\frac{Y_1 - y}{y} = \frac{-0.9598 - (-0.4)}{-0.4} = 1.3995$$

- c) Propose a time step h to obtain an approximation with a relative error three orders of magnitude smaller.

As it is obvious in the table the order of error ratios are proportional to the 3rd power of step ratios:

steps	h	h ²	h ³	error
2	-0.500	0.250	-0.125	-0.560
4	-0.250	0.063	-0.016	-0.065
ratio	0.500	0.250	0.125	0.116

So if we need a time step which has a relative error three orders of magnitude smaller than the 2 step method, then:

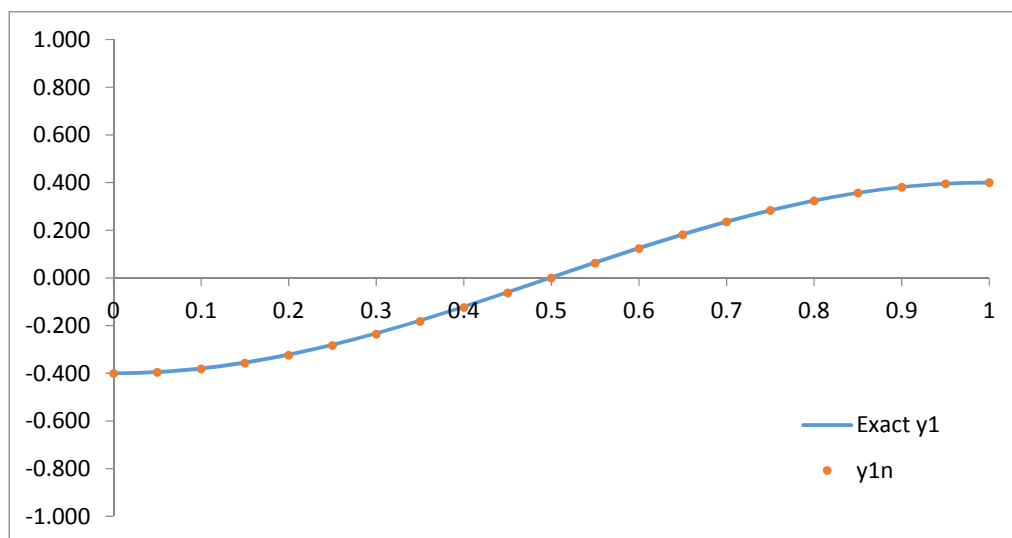
$$\text{error}_3 = \frac{\text{error}_1}{1000} \approx (-0.9598 + 0.4 = -\frac{0.5598}{1000}) \approx 0.0006$$

$$h^3 = \frac{h^3}{1000} = \frac{0.125}{1000} \rightarrow h_3 \approx 0.05$$

And using the coded program for this method, we could check if the step numbers are enough for gaining the required error or not. As it is marked we would reach to (-0.4006) with the appropriate error margin.

h=-0.05 [20 steps]										
xn	y.1	k1	y.2	k2	y1n	y.1	k1	y.2	k2	y2n
1					0.4000					0.0000
0.95	0.000	0.000	0.196	0.196	0.3951	0.400	-3.920	0.400	-3.920	0.1960
0.9	0.196	0.196	0.390	0.390	0.3805	0.395	-3.872	0.385	-3.776	0.3872
0.85	0.387	0.387	0.574	0.574	0.3564	0.380	-3.729	0.361	-3.539	0.5689
0.8	0.569	0.569	0.744	0.744	0.3236	0.356	-3.493	0.328	-3.214	0.7366
0.75	0.737	0.737	0.895	0.895	0.2828	0.324	-3.172	0.287	-2.811	0.8861
0.7	0.886	0.886	1.025	1.025	0.2351	0.283	-2.772	0.239	-2.338	1.0139
0.65	1.014	1.014	1.129	1.129	0.1815	0.235	-2.304	0.184	-1.807	1.1166
0.6	1.117	1.117	1.206	1.206	0.1234	0.181	-1.779	0.126	-1.231	1.1919
0.55	1.192	1.192	1.252	1.252	0.0623	0.123	-1.210	0.064	-0.626	1.2378
0.5	1.238	1.238	1.268	1.268	-0.0003	0.062	-0.611	0.000	-0.004	1.2531
0.45	1.253	1.253	1.253	1.253	-0.0630	0.000	0.003	-0.063	0.617	1.2376
0.4	1.238	1.238	1.207	1.207	-0.1241	-0.063	0.617	-0.125	1.224	1.1916
0.35	1.192	1.192	1.131	1.131	-0.1821	-0.124	1.216	-0.184	1.800	1.1162
0.3	1.116	1.116	1.027	1.027	-0.2357	-0.182	1.785	-0.238	2.332	1.0133
0.25	1.013	1.013	0.898	0.898	-0.2835	-0.236	2.310	-0.286	2.807	0.8854
0.2	0.885	0.885	0.746	0.746	-0.3243	-0.283	2.778	-0.328	3.212	0.7356
0.15	0.736	0.736	0.577	0.577	-0.3571	-0.324	3.178	-0.361	3.539	0.5677
0.1	0.568	0.568	0.393	0.393	-0.3811	-0.357	3.500	-0.385	3.778	0.3858
0.05	0.386	0.386	0.199	0.199	-0.3957	-0.381	3.735	-0.400	3.924	0.1943
0	0.194	0.194	0.000	0.000	-0.4006	-0.396	3.878	-0.405	3.973	-0.0020

Finally the plot for comparing the exact solution vs. numerical method is provided. As it is appeared for the small h (=0.05) numerical prediction is doing well.



2. Consider the initial value problem:

$$\frac{dy}{dx} = y - x^2 + 1 \quad x \in (0, 1)$$

$$y(0) = 1$$

a) Solve the initial value problem using the Euler method with step $h = 0.25$.

xn	x.	y.	k	Euler		
				yn	Exact y	Error
0				1.000	1.000	0.000
0.25	0.000	1.000	0.500	1.500	1.563	0.063
0.5	0.250	1.500	0.609	2.109	2.250	0.141
0.75	0.500	2.109	0.715	2.824	3.063	0.238
1	0.750	2.824	0.815	3.640	4.000	0.360

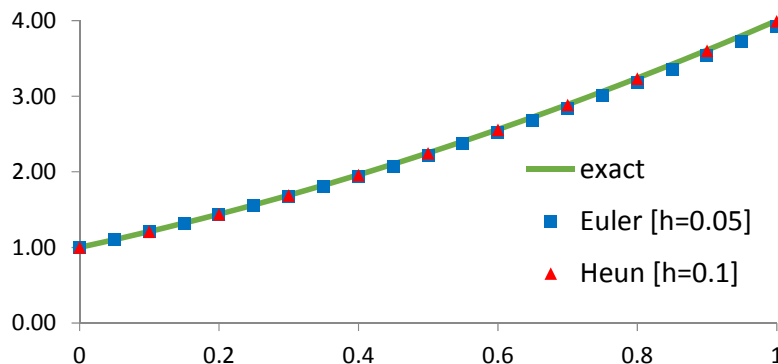
b) Compute the solution using the Heun method with a step h such that the computational cost is equivalent to the computational cost in a).

The computational cost for the Heun method is twice the Euler method so in order to maintain the computational cost we shall do the iterations in half numbers than the Euler method:

xn	x.1	x.2	k1	x.2	y.2	k2	Heun		
							yn	Exact y	Error
0							1.0000	1.000	0.0000
0.5	0.000	1.000	1.000	0.500	2.000	1.375	2.188	2.250	0.0625
1	0.500	2.188	1.469	1.000	3.656	1.828	3.836	4.000	0.1641

Finally, for the sake of comparison, the solution using both methods with much more steps but equal computation costs, are compared to the exact solution which is yielded by solving the ODE analytically:

$$y = x^2 + 2x + 1$$



4. The second-order ordinary differential equation

$$\frac{d^2y}{dx^2} + w^2y = 0$$

is defined over the domain (0, 1) and is to be solved numerically subject to the initial conditions $y(0) = 0$, $dy/dx(0) = w$, where $y(x)$ is the exact solution.

a) Reduce the above second order ODE to a system of first order ODEs.

$$\begin{aligned}y_1 &= y \rightarrow y_1' = y' = y_2 \\y_2' &= y'' = -w^2y = -w^2y_1 \\ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_2 \\ -w^2y_1 \end{bmatrix}\end{aligned}$$

b) Set $w^2 = 3$. Using the forward Euler method to integrate the system, compute the solution at $t = 1$ by hand with $n = 4$ steps. Use the Forward Euler code to check your results.

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_2 \\ -3y_1 \end{bmatrix}$$

For the forward Euler method we have:

$$Y_{i+1} = Y_i + h \cdot f(x_i, Y_i)$$

$$Y_{1_0} = 0$$

$$Y_{2_0} = w = \sqrt{3}$$

$$Y_{1_1} = Y_{1_0} + 0.25 * Y_{2_0} = 0 + 0.25 * \sqrt{3} \approx 0.433$$

$$Y_{2_1} = Y_{2_0} + 0.25 * (-3Y_{1_0}) = \sqrt{3} + 0.25 * (-3 * 0) \approx 1.732$$

$$Y_{1_2} = Y_{1_1} + 0.25 * Y_{2_1} = 0.433 + 0.25 * \sqrt{1.732} \approx 0.866$$

$$Y_{2_2} = Y_{2_1} + 0.25 * (-3Y_{1_1}) = 1.732 + 0.25 * (-3 * 0.433) \approx 1.407$$

$$Y_{1_3} = Y_{1_2} + 0.25 * Y_{2_2} = 0.866 + 0.25 * 1.407 \approx 1.218$$

$$Y_{2_3} = Y_{2_2} + 0.25 * (-3Y_{1_2}) = 1.407 + 0.25 * (-3 * 0.866) \approx 0.758$$

$$Y_{1_4} = Y_{1_3} + 0.25 * Y_{2_3} = 1.218 + 0.25 * 0.758 \approx 1.407$$

$$Y_{2_4} = Y_{2_3} + 0.25 * (-3Y_{1_3}) = 0.758 + 0.25 * (-3 * 1.218) \approx -0.156$$

Using coded program for Euler method we would have this table. As we may see all procedure is exactly the same as the hand calculated one.

		Euler				
xn	y1n	Exact y1	Error1	y2n	Exact y2	Error2
0	0.000	0.000	0.000	1.732	1.732	0.000
0.25	0.433	0.420	-0.013	1.732	1.572	-0.160
0.5	0.866	0.762	-0.104	1.407	1.122	-0.285
0.75	1.218	0.963	-0.255	0.758	0.465	-0.293
1	1.407	0.987	-0.420	-0.156	-0.278	-0.122

For having an estimate of errors, we have the exact values from the analytical solution of the ODE:

$$y = C_2 \sin(\sqrt{3}x) + C_1 \cos(\sqrt{3}x)$$

$$y(0) = 0 \rightarrow c_1 = 0$$

$$y' = \sqrt{3}C_2 \cos(\sqrt{3}x)$$

$$y'(0) = \sqrt{3} \rightarrow c_2 = 1$$

$$y = \sin(\sqrt{3}x)$$

c) Using the Matlab code, compute the solution using $n = 8$ steps. Use these solution values to estimate the step size required to obtain a numerical solution with three significant digits. Try your new step size.

Using coded program for Euler method we would have this table:

		Euler					
n	xn	y1n	Exact y1	Error1	y2n	Exact y2	Error2
0	0	0.000	0.000	0.000	1.732	1.732	0.000
1	0.125	0.217	0.215	-0.002	1.732	1.692	-0.040
2	0.25	0.433	0.420	-0.013	1.651	1.572	-0.079
3	0.375	0.639	0.605	-0.035	1.488	1.379	-0.109
4	0.5	0.825	0.762	-0.064	1.249	1.122	-0.127
5	0.625	0.982	0.883	-0.098	0.939	0.812	-0.127
6	0.75	1.099	0.963	-0.136	0.571	0.465	-0.106
7	0.875	1.170	0.998	-0.172	0.159	0.096	-0.063
8	1	1.190	0.987	-0.203	-0.280	-0.278	0.002

$$\frac{\text{error2}}{\text{error1}}(\text{at } x = 1) = 0.4834$$

Our demanded error would be (-0.0001) in order to have three significant digits in the numerical solution.

$$\frac{\text{error3}}{\text{error1}}(\text{at } x = 1) = 0.000238$$

As we know in the Euler method the local truncation error is proportional to h^2 . On the other hand over a fixed n-interval in which we want to solve the ODE the number of steps is proportional to $1/h$. Hence the total (global) error would be proportional to $1/h \cdot h^2 = h$. So required h for obtaining error3 would be around $0.000238 \cdot h(0.25) = 0.00006$. After using this small h the numerical solution using coded program would be as follows:

		Euler			
n	xn	y1n	Exact y1	Error1	y2n
0	0	0.000	0.000	0.000	1.732
1	0.00006	0.000	0.000	0.000	1.732
2	0.00012	0.000	0.000	0.000	1.732
3	0.00018	0.000	0.000	0.000	1.732
4	0.00024	0.000	0.000	0.000	1.732
5	0.0003	0.001	0.001	0.000	1.732
6	0.00036	0.001	0.001	0.000	1.732
7	0.00042	0.001	0.001	0.000	1.732
8	0.00048	0.001	0.001	0.000	1.732
9	0.00054	0.001	0.001	0.000	1.732
10	0.0006	0.001	0.001	0.000	1.732
...	...				
...	...				
16660	0.9996	0.987	0.987	0.000	-0.277
16661	0.99966	0.987	0.987	0.000	-0.277
16662	0.99972	0.987	0.987	0.000	-0.277
16663	0.99978	0.987	0.987	0.000	-0.277
16664	0.99984	0.987	0.987	0.000	-0.278
16665	0.9999	0.987	0.987	0.000	-0.278
16666	0.99996	0.987	0.987	-0.0001	-0.278
16667	1.00002	0.9871	0.987	-0.0001	-0.278

As we may check the final numerical solution is 0.9871 which compared to the exact solution (0.9870) has 3 significant digits.