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Numerical Methods for Partial Differential Equation.
ODE
Exercise – 2

Ans – 1

a) $$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

let $\frac{d\theta}{dt} = z$ ...... eq ① $\theta(1) = 0.4$ rad

$= f_1(t, \theta, z)$

$\therefore \frac{dz}{dt} + \frac{g}{L}\theta = 0$

$\frac{dz}{dt} = -\frac{g}{L}\theta$ ...... eq ② $z(1) = 0$ rad/s

$= f_2(t, \theta, z)$

2nd order runge Kutta method

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}K_1^{\theta} + \frac{1}{2}K_2^{\theta}\right)h$$

$$z_{i+1} = z_i + \left(\frac{1}{2}K_1^{z} + \frac{1}{2}K_2^{z}\right)h$$

where,

$$K_1^{\theta} = f_1(t_i, \theta_i, z_i)$$

$$K_2^{\theta} = f_1(t_i + h, \theta_i + K_1^{\theta}h, z_i + K_1^{z}h)$$

$$K_1^{z} = f_2(t_i, \theta_i, z_i)$$

$$K_2^{z} = f_2(t_i + h, \theta_i + K_1^{\theta}h, z_i + K_1^{z}h)$$

With 2 time steps

$i=0,\ h=-0.5,\ \theta_0=0.4,\ t_0=1,\ z_0=0$ rad/s

$$K_1^\theta = f_1(t_0, \theta_0, z_0) = f_1(1, 0.4, 0) = 0$$

$$K_1^z = f_2(t_0, \theta_0, z_0) = f_2(1, 0.4, 0) = -3.92$$

$$K_2^\theta = f_1(t_0+h,\ \theta_0+K_1^\theta h,\ z_0+K_1^z h) = f_1(0.5, 0.4, 1.96) = 1.96$$

$$K_2^z = f_2(t_0+h,\ \theta_0+K_1^\theta h,\ z_0+K_1^z h) = f_2(0.5, 0.4, 1.96) = -3.92$$

$$\therefore \theta_1 = \theta_0 + \left(\tfrac{1}{2}K_1^\theta + \tfrac{1}{2}K_2^\theta\right)h = 0.4 + \left(\tfrac{1}{2}\times 0 + \tfrac{1}{2}\times 1.96\right)\times -0.5 = -0.09 \text{ rad}$$

$$z_1 = z_0 + \left(\tfrac{1}{2}K_1^z + \tfrac{1}{2}K_2^z\right)h = 0 + \left(\tfrac{1}{2}\times -3.92 + \tfrac{1}{2}\times -3.92\right)\times -0.5 = -1.96$$

When,

$i=1,\ t_1=0.5,\ h=-0.5,\ \theta_1=-0.09\ \ z_1=1.96$

$k_1^\theta = f_1(t_1, \theta_1, z_1)$

$= f_1(0.5, -0.09, 1.96)$

$= 1.96$

$k_1^z = f_2(t_1, \theta_1, z_1)$

$= f_2(0.5, -0.09, 1.96)$

$= 0.882$

$k_2^\theta = f_1(t_1+h, \theta_1 + k_1^\theta h, z_1 + k_1^z h)$

$= f_1(0, -1.97, 1.519)$

$= 1.519$

$k_2^z = f_2(t_1+h, \theta_1 + k_1^\theta h, z_1 + k_1^z h)$

$= f_2(0, -1.97, 1.519)$

$= 19.306$

$\theta_2 = \theta_1 + \left(\frac{1}{2}k_1^\theta + \frac{1}{2}k_2^\theta\right)h$

$= -0.09 + \left(\frac{1}{2}\times 1.96 + \frac{1}{2}\times 1.519\right)\times -0.5$

$= -0.95975$

With 4 time step

$i=0$, $h=-0.25$, $t_0=1$, $\theta_0=0.4$ rad $\cancel{ls}$ $z_0=0$ rad/s

$K_1^\theta = f_1(1, 0.4, 0)$

$=0$

$K_1^z = f_2(1, 0.4, 0)$

$=-3.92$

$K_2^\theta = f_1(1-0.25, 0.4+0, 0+(-3.92\times-0.25))$

$= f_1(0.75, 0.4, 0.98)$

$=0.98$

$K_2^z = f_2(0.75, 0.4, 0.98)$

$=-3.92$

$\theta_1 = \theta_0 + (\frac{1}{2}K_1^\theta + \frac{1}{2}K_2^\theta)h$

$=0.4+(\frac{1}{2}\times0 + \frac{1}{2}\times0.98)\times-0.25$

$=0.2775$

$z_1 = z_0 + (\frac{1}{2}K_1^z + \frac{1}{2}K_2^z)h$

$=0+(\frac{1}{2}\times-3.92 + \frac{1}{2}\times-3.92)\times-0.25$

$=0.98$

when,

$i=1,\quad t_1=0.75,\quad h=-0.25,\quad \theta_1=0.2775\quad z_1=0.98$

$K_1^{\theta}=f_1(0.75,\ 0.2775,\ 0.98)$

$=0.98$

$K_1^{z}=f_2(0.75,\ 0.2775,\ 0.98)$

$=-2.7195$

$K_2^{\theta}=f_1(0.5,\ 0.0325,\ 1.659875)$

$=1.659875$

$K_2^{z}=f_2(0.5,\ 0.0325,\ 1.659875)$

$=-0.3185$

$\theta_2=\theta_1+\left(\frac{1}{2}K_1^{\theta}+\frac{1}{2}K_2^{\theta}\right)h$

$=0.2775+\left(\frac{1}{2}0.98+\frac{1}{2}1.659875\right)\times-0.25$

$=-0.05248$

$z_2=z_1+\left(\frac{1}{2}K_1^{z}+\frac{1}{2}K_2^{z}\right)h$

$=0.98+\left(\frac{1}{2}\times-2.7195+\frac{1}{2}\times0.3185\right)\times-0.25$

$=1.35975$

When, $i=2$, $h=-0.25$, $t_2=0.5$, $\theta_2=-0.05248$, $z_2=1.35975$

$$K_1^\theta = f_1(0.5, -0.05248, 1.35975)$$

$$= 1.35975$$

$$K_1^z = f_2(0.5, -0.05248, 1.35975)$$

$$= 0.514304$$

$$K_2^\theta = f_1(0.25, -0.3924175, 1.231174)$$

$$= 1.231174$$

$$K_2^z = f_2(0.25, -0.3924175, 1.231174)$$

$$= 3.8456915$$

$$\theta_3 = \theta_2 + \left(\tfrac{1}{2}K_1^\theta + \tfrac{1}{2}K_2^\theta\right)h$$

$$= -0.05248 + \left(\tfrac{1}{2}\times 1.35975 + \tfrac{1}{2}\times 1.231174\right)\times -0.25$$

$$= -0.37635$$

$$z_3 = z_2 + \left(\tfrac{1}{2}K_1^z + \tfrac{1}{2}K_2^z\right)h$$

$$= 1.35975 + \left(\tfrac{1}{2}\times 0.514304 + \tfrac{1}{2}\times 3.8456919\right)\times -0.25$$

$$= 0.8147505$$

when,

$i=3, \quad t_3 = +0.25 \quad h = -0.25, \ \theta_3 = -0.37635, \ z_3 = 0.8147505$

$K_1^{\theta} = f_1(0.25, -0.37635, 0.8147505)$

$= 0.8147505$

$K_1^{z} = f_2(0.25, -0.37635, 0.8147505)$

$= 3.68823$

$K_2^{\theta} = f_1(0, -0.580037, -0.107307)$

$= -0.107307$

$\theta_4 = \theta_3 + \left(\frac{1}{2}K_1^{\theta} + \frac{1}{2}K_2^{\theta}\right)h$

$= -0.37635 + \left(\frac{1}{2} \times 0.8147505 + \frac{1}{2} \times -0.107307\right) \times -0.25$

$= -0.46478$

b) 2-step $\rightarrow$ $\theta_2 = -0.95975$

4-step $\rightarrow$ $\theta_4 = -0.46478$

$$\text{Relative error} = \frac{\theta_2 - \theta_4}{\theta_4}$$

$$= \frac{-0.95975 - (-0.46478)}{-0.46478} \times 100$$

$= 106.496\%$

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c) Using,

$$h^* = \left(\frac{tol}{E_n}\right)^{1/p+1} h$$

where,

$$\frac{tol}{E_n} = 10^{-3}$$

$$h = 0.5$$

$$p = 2$$

$$\therefore h^* = (10^{-3})^{\frac{1}{3}} \times 0.5$$

$$= 0.05$$

So, time step $= \frac{1-0}{0.05} = 20$ steps

Ans - 2

a) $\frac{dy}{dx} = y - x^2 + 1$

$y(0) = 1$

$h = 0.25$

$y_1 = y_0 + hf(x_0 y_0)$ when, $x_0 = 0$, $y_0 = 1$

$y_1 = 1 + 0.25(1 - 0^2 + 1)$

$= 1.5$

$y_2 = y_1 + 0.25 f(x_1 y_1)$ where, $x_1 = 0.25$, $y_1 = 1.5$

$y_2 = 1.5 + 0.25 f(0.25, 1.5)$

$= 2.109375$

$y_3 = y_2 + 0.25 f(x_2 y_2)$ where, $x_2 = 0.5$, $y_2 = 2.109375$

$y_3 = 2.109375 + 0.25 f(0.5, 2.109375)$

$= 2.82421875$

$y_4 = y_3 + 0.25 f(x_3 y_3)$ where, $x_3 = 0.75$ $y_3 = 2.82421875$

$y_4 = 2.82421875 + 0.25 f(0.75, 2.82421875)$

$= 3.639648438$ //

b) Heun

$$Y_{i+1} = Y_i + \frac{h}{2}(K_1 + K_2)$$

$$\frac{dy}{dx} = y - x^2 + 1$$

$K_1 = f(x_i, y_i)$

$K_2 = f(x_i + h, y_i + hK_1)$

$i=0, \quad x_0 = 0, \quad y_0 = 1, \quad h = 0.25$

$K_1 = f_1(x_0\ y_0)$

$= f_1(0, 1)$

$= 2$

$K_2 = f_1(x_0 + h, y_0 + K_1 h)$

$= f_1(0 + 0.25, 1 + 2 \times 0.25)$

$= f_1(0.25, 1.5)$

$= 2.4375$

$$Y_1 = Y_0 + \frac{0.25}{2}(2 + 2.4375)$$

$$= 1 + \frac{0.25}{2}(2 + 2.4375)$$

$= 1.5547$

when,

$x_1 = 0.25 \quad y_1 = 1.5547 \quad h = 0.25$

$K_1 = f_1(0.25, 1.5547)$

$= 2.4922$

$K_2 = f(0.5, 2.17775)$

$= 2.92775$

$y_2 = y_1 + \frac{h}{2}(K_1 + K_2)$

$= 1.5547 + \frac{0.25}{2}(2.4922 + 2.92775)$

$= 2.2322$

when,

$x_2 = 0.5, \quad y_2 = 2.2322 \quad h = 0.25$

$K_1 = f(0.5, 2.2322)$

$= 2.9822$

$K_2 = f(0.75, 2.97775)$

$= 3.41525$

$y_3 = y_2 + \frac{h}{2}(K_1 + K_2)$

$= 2.2322 + \frac{0.25}{2}(2.9822 + 3.41525)$

$= 3.0319$

when,

$x_3 = 0.75 \quad y_3 = 3.0319 \quad h = 0.25$

$K_1 = f_3(0.75,\ 3.0319)$

$= 3.4694$

$K_2 = f_3(0.75 + 0.25,\ 3.0319 + 3.4694 \times 0.25)$

$= f_3(1,\ 3.89925)$

$= 3.89925$

$y_4 = 3.0329 + \frac{0.25}{2}(3.4694 + 3.89925)$

$= 3.95298$

Ans-3

a) Using taylor series expansion

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2}$$

$$\frac{dy}{dx} = \frac{y_{i+1} - y_i}{h} - \frac{h}{2!}\frac{d^2y}{dx^2}$$

$$\frac{dy}{dx} = \frac{y_{i+1} - y_i}{h} - \tau(h)$$

where, $\tau(h) = O(h)$ is the truncation error.

For the method to be consistent it has to verify

$$\max \tau(h) \to 0 \quad \text{when} \quad h \to 0$$

In our above case, when $h \to 0$, the $\max \tau(h) \to 0$,
so, it is consistent.

b) Backward Euler method,

$$y_i = y_{i+1} - h\frac{dy}{dx} + O(h^2)$$

$$\frac{dy}{dx} = \frac{y_{i+1} - y_i}{h} + \tau(h)$$

$$\therefore y_{i+1} = y_i + hf(x_{i+1}, y_{i+1}) + h\tau_i(h)$$

neglecting the error,

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$$

c) Stability limits for forward Euler method:-

$$\frac{dy}{dx} = -\lambda y$$

$$y_{i+1} = y_i - h\lambda y_i$$

$$y_{i+1} = G y_i \quad \text{with} \quad G = 1 - h\lambda$$

The scheme is stable in $|G| < 1$

$$\therefore |1 - h\lambda| < 1$$

$$h\lambda > 0$$
$$h\lambda < 2$$

$$\therefore \boxed{2 > h\lambda > 0}$$

Stability limits for backward Euler method is:-

$$\frac{dy}{dx} = -\lambda y$$

$$\therefore y_{i+1} = y_i - h\lambda y_{i+1}$$

$$(1 + h\lambda) y_{i+1} = y_i$$

$$y_{i+1} = G y_i \quad \text{with} \quad G = \frac{1}{1 + h\lambda}$$

Scheme is stable is $|1 + h\lambda| > 1$

$$\boxed{h\lambda > 0 \qquad h\lambda < -2}$$ ← unconditionally stable

d) $\frac{dy}{dx} = -25y^{3.5}$

$y(0) = 1 \qquad h = \frac{1}{10}$

when $i=0$,

$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$

$y_1 = y_0 + hf(x_1, y_1)$

$y_1 = 1 + \frac{1}{10} f(x, y_1)$

$y_1 = 1 + \frac{1}{10}(-25y_1^{3.5})$

$2.5y_1^{3.5} + y_1 - 1 = 0$

$f(y) = 2.5y_1^{3.5} + y_1 - 1 = 0$

$f'(y) = 8.75y_1^{2.5} + 1$

Applying newton method with initial guess $(y_1)_0 = 1$

$(y_1)_1 = (y_1)_0 - \frac{f(y)}{f'(y)}$

$= 1 - \frac{2.5}{9.75}$

$= 0.74358$

$(y_1)_2 = (y_1)_1 - \frac{f(y)}{f'(y)}$

$= 0.74358 - \frac{0.629892}{5.1718367}$

$= 0.62178$

when,

$i=2, \quad y_1 = 0.62178 \quad h = \frac{1}{10}$

$$y_2 = y_1 + hf(x_2\, y_2)$$

$$y_2 = 0.62178 + \frac{1}{10}(-25y_2^{3.5})$$

$$y_2 = 0.62178 - 2.5y_2^{3.5}$$

$$2.5y_2^{3.5} + y_2 - 0.62178 = 0$$

$$f(y_2) = 2.5y_2^{3.5} + y_2 - 0.62178 = 0$$

$$f'(y_2) = 8.75y_2^{2.5} + 1$$

Using Newton Method with initial guess $(y_2)_0 = 0.62178$

$$(y_2)_1 = (y_2)_0 - \frac{f(y_2)}{f'(y_2)}$$

$$= 0.62178 - \frac{0.47388}{3.66747}$$

$$= 0.49257$$

$$(y_2)_2 = (y_2)_1 - \frac{f(y_2)}{f'(y_2)}$$

$$= 0.49257 - \frac{0.0804801}{2.48997}$$

$$= 0.4602$$

c) $\frac{dy}{dx} = -25y^{3.5}$

$y(0) = 1, \quad x_0 = 0 \quad y_0 = 1 \quad h = \frac{1}{10}$

$y_{i+1} = y_i + hf(x_i, y_i)$

when $i = 0$,

$y_1 = y_0 + hf(x_0 y_0)$

$= 1 + \frac{1}{10} f(0, 1)$

$= 1 - 2.5$

$= -1.5$

when $i = 1$,

$y_2 = y_1 + hf(x_1, y_1)$

$= -1.5 + \frac{1}{10}\left(\frac{1}{10}, -1.5\right)$

$= -1.5 + \frac{1}{10}\left(-25 \times (-1.5)^{3.5}\right)$

The solution involves complex number, which is adding up, thus increasing the value of y, which means the solution is unstable.