Numerical Methods for PDE
Homework – Finite Difference
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2) Let consider the differential equation.

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad x \in (0,1)$$

$$U(x,0) = \sin(2\pi x) \qquad u(0,t) = u(1,t)$$

a). Backward in time: $\left.\frac{\partial u}{\partial t}\right|_i^{n+1} = \frac{U_i^{n+1} - U_i^n}{\Delta t} + O(\Delta t)$

· Backward in space: $\left.\frac{\partial u}{\partial x}\right|_i^{n+1} = \frac{U_i^{n+1} - U_{i-1}^{n+1}}{\Delta x} + O(\Delta x)$

$$U_i^{n+1} + c\,u_i^{n+1} - c\,u_{i-1}^{n+1} = u_i^n$$

$$u_i^{n+1}(1+c) - c\,u_{i-1}^{n+1} = u_i^n$$

· there is no particular reason for chosen backward approximatio in space, I belive a Forward approximation can be used also. However, backward approximation guarantees that the A matrix will be diagonally dominant, which is an advantege in case an iterative method is to be used for solution.

b) Since we have periodic boundary conditions

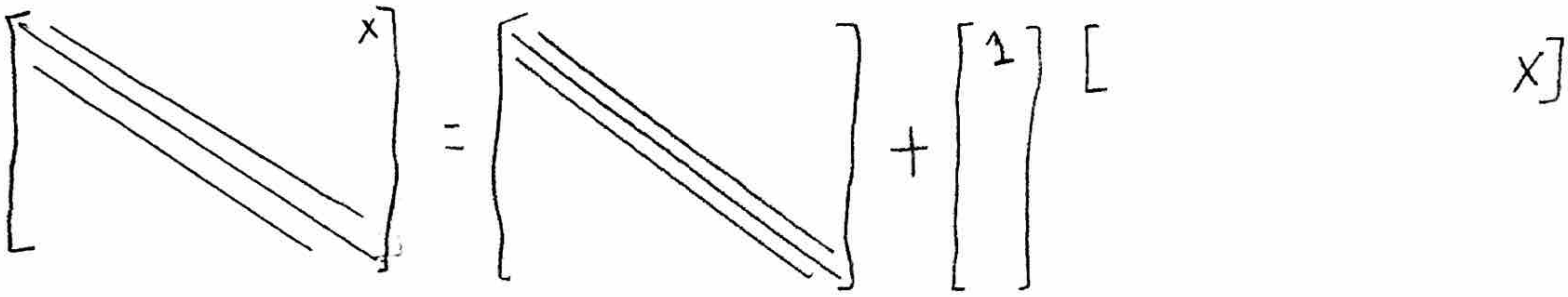
$$U_0^n = U_{m+1}^n$$

$$\begin{bmatrix} (1+C) & 0 & 0 & \cdots & C \\ -C & (1+C) & 0 & \cdots & 0 \\ 0 & -C & (1+C) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & & -C & (1+C) \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \\ U_3^n \\ \vdots \\ U_{m+1}^n \end{bmatrix} = \begin{bmatrix} U_1^n \\ U_2^n \\ U_3^n \\ \vdots \\ U_{m+1}^n \end{bmatrix}$$

c) • a direct method for the solution of the linear system of equation. would be <u>Sherman-Morrison</u> method

• an iterative method for the solution of the linear system of equation is <u>Gauss-Seidal method</u>

b) $A = A^{*} + UV^{T}$



4) $\dfrac{\partial U}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2} + \sigma u \qquad x \in (0,1),\ t > 0$

$$u(0,t) = 0 \qquad \frac{\partial u}{\partial x} = (1,t) = 0$$

$$U(x,0) = \begin{cases} 0 & \text{for } x < 1/4 \\ 4x-1 & \text{for } 1/4 \leq x < 1/2 \\ -4x+3 & \text{for } 1/2 \leq x < 3/4 \\ 0 & \text{for } 3/4 \leq x \end{cases}$$

A) an explicit finite difference scheme

- forward in time : $\left.\dfrac{\partial u}{\partial t}\right|_i^n = \dfrac{u_i^{n+1} - u_i^n}{\Delta t}$

- Centered in space: $\left.\dfrac{\partial^2 u_x}{\partial x^2}\right|_i^{n-1} = \dfrac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2}$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} + \sigma u_i^n$$

$$u_i^{n+1} = C\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right) + u_i^n(1 + \Delta t\sigma) \qquad n = 1 \ldots\ldots t$$

$$i = 1 \ldots\ldots m$$

since $\dfrac{\partial u}{\partial x}(1,t) = 0 \Rightarrow U_{m+1}^n = U_{m-1}^n$

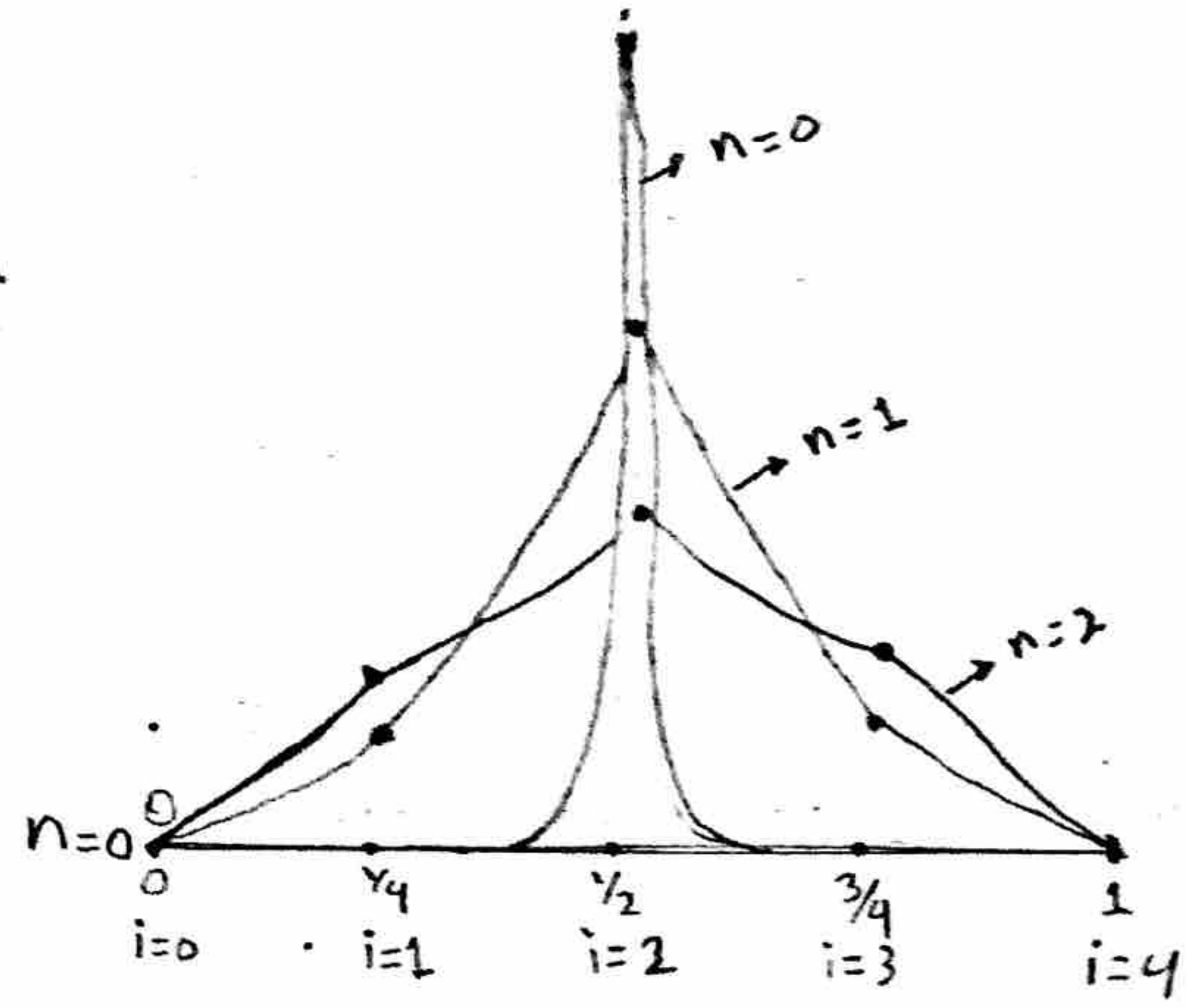
$$C = \nu \frac{\Delta t}{\Delta x^2}$$

$$u_i^0 = \begin{cases} 0 & i\Delta x < 1/2 \\ 4(i\Delta x) - 1 & 1/2 \leq i\Delta x \leq 1/2 \\ -4(i\Delta x) + 3 & 1/2 \leq i\Delta x \leq 3/4 \\ 0 & i\Delta x \geq 3/4 \end{cases}$$

B). for $\sigma = 0$, the scheme will be the explicit method (FTCS) for 1D parabolic equation

- for $\nu = 0$, the scheme will be Euler Forward approximation For OD

c) $\nu = 0.1$, $\sigma = -0.1$, $\Delta x = 0.25$ and $\Delta t = 0.1$




→ $n = 1$

$i=1$ $\quad U_1^1 = 0.16(0-2(0)+1)+0 = 0.16$

$i=2$ $\quad U_2^1 = 0.16(0-2(1)+0)+(1)(1-(0.1)(0.25)) =$
$= 0.655$

$i=3$ $\quad U_3^1 = 0.16(1-2(0)+0)+0 = 0.16$

$i=4$ $\quad U_4^1 = 0.16(0-2(0)+0)+0 = 0$

→ $n = 2$

$i=1$ $\quad U_1^2 = 0.16(0-2(.16)+0.655)+(0.16)(1-(0.1)(0.25)) = 0.2096$

$i=2$ $\quad U_2^2 = 0.16(0.16-2(0.655)+0.16)+(0.655)(1-(0.1)(0.25)) = 0.480225$

$i=3$ $\quad U_3^2 = 0.16(0.655-2(0.16)+0)+(0.16)(1-(0.1)(0.25)) = 0.2096$

$i=4$ $\quad U_4^2 = 0.16(0.16-2(0)+0.16)+(0)(1-(0.1)(0.25)) = 4.096 * 10^{-3}$

From the profile of the solution, it can be said that this is a reasonable solution of diffusion-reaction problem, the overall look of the profile look like a solution of parabolic equation, while the unexpected increases in the nodes 1, 3 at $n=2$, highlights the reaction effect.

d) An implicit finite different scheme

- backward in time: $\left.\frac{\partial u}{\partial t}\right|_i^{n+1} = \frac{U_i^{n+1} - U_i^n}{\Delta t}$
- Centered in space: $\left.\frac{\partial^2 u}{\partial x^2}\right|_i^{n+1} = \frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{\Delta x^2}$

$$U_i^{n+1} - U_i^n = \nu\frac{\Delta t}{\Delta x^2}\left(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}\right) + \sigma U_i^{n+1}$$

$$-CU_{i-1}^{n+1} + (2C-\sigma+1)U_i^{n+1} - CU_{i+1}^{n+1} = U_i^n \qquad i=1.....m$$

$$n=1.....m$$

$$U_i^0 = \begin{cases} 0 & i\Delta x < \tfrac{1}{2} \\ 4(i\Delta x)-1 & \tfrac{1}{2} \leq i\Delta x \leq \tfrac{1}{2} \\ -4(i\Delta x)+3 & \tfrac{1}{2} \leq i\Delta x \leq \tfrac{3}{4} \\ 0 & i\Delta x \geq \tfrac{3}{4} \end{cases}$$

$$U_{m+1}^n = U_{m-1}^n \qquad C = \frac{\nu\Delta t}{\Delta x^2}$$

$$\begin{bmatrix} (2C-\sigma+1) & -C & 0 & \cdots & 0 \\ -C & (2C-\sigma+1) & -C & \cdots & 0 \\ 0 & -C & (2C-\sigma+1) & \ddots & -C \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \boxed{-2C} & & (2-\sigma+1) \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_m^{n+1} \end{bmatrix} = \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_m^n \end{bmatrix} + \begin{bmatrix} CU_0^n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$