Numerical Methods for Partial Differential Equations

Ordinary Differential Equations

1. The motion of a non-frictional pendulum is governed by the Ordinary Differential Equation (ODE)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where θ is the angular displacement, $L = 1$ m is the pendulum length and the gravity acceleration is $g = 9.8$ m/s^2.

The position and velocity at time $t = 1$ s are known:

$$\theta(1) = 0.4 \text{ rad} \qquad ; \qquad \frac{\mathrm{d}\theta}{\mathrm{dt}}(1) = 0 \text{ rad/s}$$

a) Solve the initial boundary value problem in the interval $(0, 1)$ using a second-order Runge-Kutta method to determine the initial position at $t = 0$ s, with 2 and 4 time steps.

b) Using the approximations obtained in *a)*, compute an approximation of the relative error in the solution computed with 2 steps.

c) Propose a time step h to obtain an approximation with a relative error three orders of magnitude smaller.

---

2. Consider the initial value problem

$$\begin{aligned} \frac{dy}{dx} &= y - x^2 + 1 \qquad x \in (0, 1) \\ y(0) &= 1 \end{aligned}$$

a) Solve the initial value problem using the Euler method with step $h = 0.25$.

b) Compute the solution using the Heun method with a step h such that the computational cost is equivalent to the computational cost in *a)*.

Note that the analytical solution of the initial value problem is a second degree polynomial.

c) Compute the pure interpolation polynomial that fits the results in *b)*.

d) Which approximation criterion do you recommend to fit the results obtained in *a)*? Compute the polynomial approximation with the proposed criterion and compare the results with the polynomial obtained in *c)*.

---

3. The ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

is defined over the domain (0,1), and is to be solved numerically subject to the initial condition $y(0) = 1$, where $y(x)$ is the exact solution. The forward Euler method for integrating the above differential equation is written as

$$Y_{i+1} = Y_i + hf(x_i, Y_i)$$

where Y_i denotes the discrete solution at node i, with position x_i, of a uniform grid of nodes of constant grid interval size h and $x_{i+1} = x_i + h$.

a) Using a Taylor series expansion, deduce the leading truncation error of the scheme. Is the method consistent? Explain your answer.

b) State the backward Euler method for integrating the above differential equation where $f(x, y)$ is a general non-linear function of x and y.

c) Deduce the stability limits of the respective forward Euler method and backward Euler method for the model equation $dy/dx = -\lambda y$ where λ is a positive real constant.

d) Use the backward Euler method to compute the numerical solution of the ordinary differential equation

$$\frac{dy}{dx} = -25y^{3.5}$$

with initial condition $y(0) = 1$, by hand for two steps with grid interval size $h = 1/10$. (Use 2 Newton iterations per step for this calculation.)

e) Use the forward Euler method to compute the numerical solution of the above ordinary differential equation with same initial condition by hand for two steps with grid interval size h=1/10.

f) The analytical solution is

$$y(x) = \left(\frac{125x + 2}{2}\right)^{-2/5}$$

Using Matlab codes, indicate the maximum stable interval size possible for forward Euler method from the following; h=1/10, h=1/15, h=1/30, h=1/45, h=1/90. How does your choice compare with the stability condition?

---

4. The second-order ordinary differential equation

$$\frac{d^2y}{dx} + \omega^2 y = 0$$

is defined over the domain $(0, 1)$, and is to be solved numerically subject to the initial conditions $y(0) = 0$, $dy/dx(0) = \omega$, where $y(x)$ is the exact solution.

a) Reduce the above second order ODE to a system of first order ODEs.

b) Set $\omega^2 = 3$. Using the forward Euler method to integrate the system, compute the solution at $t = 1$ by hand with $n = 4$ steps. Use the Forward Euler code to check your results.

c) Using the Matlab code, compute the solution using $n = 8$ steps. Use these solution values to estimate the step size required to obtain a numerical solution with three significative digits. Try your new step size.

Numerical Methods for Partial differential equations

Pol Goma

ODE exercises

1.

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \qquad l = 1\,m \quad g = 9.8\,m/s^2 \qquad \theta(1) = 0.4\ rad \qquad \frac{d\theta}{dt}(1) = 0\ rad/s$$

a) Initial position at t = 0 s? interval (0, 1)

$$y_1 = \theta \qquad \frac{dy_1}{dt} = \frac{d\theta}{dt} = y_2$$

$$y_2 = \frac{d\theta}{dt} \qquad \frac{dy_2}{dt} = \frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta = -\frac{g}{l}y_1$$

$$\rightarrow \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \frac{d\theta}{dt} \end{bmatrix} \qquad f = \begin{bmatrix} y_2 \\ -\frac{g}{l} y_1 \end{bmatrix} = \begin{bmatrix} \frac{d\theta}{dt} \\ -9.8\,\theta \end{bmatrix}$$

2 time steps

t_0: $t=1$, t_1: $t=0.5$, t_2: $t=0$

$$h = \frac{0-1}{2} = -1/2$$

Heun Method (2nd order Runge-Kutta method)

$$Y^*_{i+1} = Y_i + h f(x_i, Y_i)$$

$$Y_{i+1} = Y_i + \frac{h}{2}\left[f(x_i, Y_i) + f(x_i, Y^*_{i+1}) \right]$$

$\Rightarrow i = 0$

$$Y^*_1 = Y_0 - \frac{1}{2} f(t_0, Y_0) = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ -9.8 \cdot 0.4 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 49/25 \end{bmatrix}$$

$$Y_1 = Y_0 - \frac{1}{4}\left[f(t_0, Y_0) + f(t_0, Y^*_1) \right] = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - \frac{1}{4}\left(\begin{bmatrix} 0 \\ -98/25 \end{bmatrix} + \begin{bmatrix} 49/25 \\ -9.8 \cdot 0.4 \end{bmatrix} \right) = \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix}$$

$\Rightarrow i = 1$

$$Y^*_2 = Y_1 - \frac{1}{2} f(t_1, Y_1) = \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1.96 \\ -9.8 \cdot (-0.09) \end{bmatrix} = \begin{bmatrix} -1.07 \\ 1.519 \end{bmatrix}$$

$$Y_2 = Y_1 - \frac{1}{4}\left[f(t_1, Y_2) + f(t_1, Y^*_2) \right] = \begin{bmatrix} -0.09 \\ 1.96 \end{bmatrix} - \frac{1}{4}\left(\begin{bmatrix} 1.96 \\ -9.8 \cdot (-0.09) \end{bmatrix} + \begin{bmatrix} 1.519 \\ -9.8(-1.07) \end{bmatrix} \right) = \begin{bmatrix} -0.9597 \\ -0.882 \end{bmatrix}$$

$$\boxed{\theta(0) = -0.9597\ rad}$$

4 time steps

t_0: $t=1$, t_1: $t=0.75$, t_2: $t=0.5$, t_3: $t=0.25$, t_4: $t=0$

$$h = \frac{0-1}{4} = -1/4$$

$\Rightarrow i = 0$

$$Y^*_1 = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0 \\ -98/25 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.98 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - \frac{1}{8}\left(\begin{bmatrix} 0 \\ -98/25 \end{bmatrix} + \begin{bmatrix} 0.98 \\ -9.8 \cdot 0.4 \end{bmatrix} \right) = \begin{bmatrix} 0.277 \\ 0.98 \end{bmatrix}$$

$\Rightarrow i = 1$

$$Y^*_2 = \begin{bmatrix} 0.277 \\ 0.98 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0.98 \\ -9.8 \cdot 0.277 \end{bmatrix} = \begin{bmatrix} 0.032 \\ 1.658 \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} 0.277 \\ 0.98 \end{bmatrix} - \frac{1}{8}\left(\begin{bmatrix} 0.98 \\ -9.8 \cdot 0.277 \end{bmatrix} + \begin{bmatrix} 1.658 \\ -9.8 \cdot 0.032 \end{bmatrix} \right) = \begin{bmatrix} -0.052 \\ 1.358 \end{bmatrix}$$

$\Rightarrow i = 2$

$$Y_3^* = \begin{bmatrix} -0.052 \\ 1.358 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1.358 \\ -9.8\cdot(-0.052) \end{bmatrix} = \begin{bmatrix} -0.3915 \\ 1.2306 \end{bmatrix}$$

$$Y_3 = \begin{bmatrix} -0.052 \\ 1.358 \end{bmatrix} - \frac{1}{8}\left(\begin{bmatrix} 1.358 \\ -9.8\cdot(-0.052) \end{bmatrix} + \begin{bmatrix} 1.2306 \\ -9.8\cdot(-0.3915) \end{bmatrix}\right) = \begin{bmatrix} -0.3755 \\ 0.814 \end{bmatrix}$$

$\Rightarrow i = 3$

$$Y_4^* = \begin{bmatrix} -0.3755 \\ 0.814 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0.814 \\ -9.8\cdot(-0.3755) \end{bmatrix} = \begin{bmatrix} -0.579 \\ -0.106 \end{bmatrix}$$

$$Y_4 = \begin{bmatrix} -0.3755 \\ 0.814 \end{bmatrix} - \frac{1}{8}\left(\begin{bmatrix} 0.814 \\ -9.8\cdot(-0.3755) \end{bmatrix} + \begin{bmatrix} -0.106 \\ -9.8(-0.579) \end{bmatrix}\right) = \begin{bmatrix} -0.464 \\ -0.355 \end{bmatrix}$$

$$\boxed{\theta(0) = -0.464 \text{ rad}}$$

B)

In order to compute the relative error, we first have computed the solution with 1000 time steps using matlab. we are going to use this one to compute the error we obtain using the 2 time steps approach.

$$\text{rel. error }(\%) = \left|\frac{sol_{1000\,\Delta t} - sol_{2\Delta t}}{sol_{2000\,\Delta t}}\right| \cdot 100 = \left|\frac{-0.4000-(-0.9597)}{-0.4000}\right| = 139.925\%$$

C)

$$h^* = \left(\frac{tol}{E_h}\right)^{1/p+1} h$$

p+1 → order of the method (2 in our case

$$h^* = \left(\frac{10^{-3}\cancel{E_h}}{\cancel{E_h}}\right)^{1/2} \cdot (-1/2) = -0.016 \Rightarrow \boxed{h^* = -0.016}$$

2.

$$\frac{dy}{dx} = y - x^2 + 1 \qquad x \in (0,1) \qquad y(0) = 1$$

a) $h = 0.25$

Euler Method:

y_0	y_1	y_2	y_3	y_4
0	0.25	0.5	0.75	1
x_0	x_1	x_2	x_3	x_4

$$y_{i+1} = y_i + h f(x_i, y_i)$$

$$y_1 = y_0 + 0.25 f(x_0, y_0) = 1 + 0.25(1 - 0 + 1) = 1.5$$

$$y_2 = y_1 + 0.25(1.5 - 0.25^2 + 1) = 2.109$$

$$y_3 = 2.109 + 0.25(2.109 - 0.5^2 + 1) = 2.823$$

$$\Rightarrow y_4 = 2.823 + 0.25(2.823 - 0.75^2 + 1) = 3.638$$

B) The computational cost is given by the number of times the function is evaluated. In the Euler method, the function is evaluated once in each step. However, in the Heun method, the function is evaluated twice.

This way, since we have done 4 steps with the Euler Method, we need to carry out the Heun method with 2 steps in order to maintain the computational cost.

Heun Method:

$$y^*_{i+1} = y_i + h f(x_i, y_i)$$

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_i, y^*_{i+1})]$$

$$h = \frac{1-0}{2} = 1/2$$

t_0	t_1	t_2
0	1/2	1
x_0	x_1	x_2

$\Rightarrow i = 0$:

$$y_1^* = y_0 + \frac{1}{2} f(x_0, y_0) = 1 + \frac{1}{2}(1 - 0 + 1) = 2$$

$$y_1 = y_0 + \frac{1}{4}[f(x_0, y_0) + f(x_0, y_1^*)] = 1 + \frac{1}{4}[2 + 3] = 2.25$$

$\Rightarrow i = 1$:

$$y_2^* = y_1 + \frac{1}{2} f(x_1, y_1) = 2.25 + \frac{1}{2}(2.25 - 0.5^2 + 1) = 3.75$$

$$y_2 = y_1 + \frac{1}{4}[f(x_1, y_1) + f(x_1, y_2^*)] = 2.25 + \frac{1}{4}(2.25 - 0.5^2 + 1 + 3.75 - 0.5^2 + 1) = 4.125$$

NOT NECESSARY ANYMORE

c)

3.

$\frac{dy}{dx} = f(x,y)$ in $(0,1)$ $\qquad$ $y(x)$ = exact solution

$y(0) = 1$

Forward Euler Method :

$$Y_{i+1} = Y_i + h f(x_i, Y_i)$$

Taylor Series:

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

a) Taylor expansion of y around x_i:

$$y(x_i+h) = Y_{i+1} = y(x_i) + \frac{1}{1!}\frac{dy}{dx}(x_i)(x-x_i) + O(h^2) = y(x_i) + \frac{dy}{dx}(x_i)(\cancel{x_i}+h-\cancel{x_i}) + O(h^2) = y(x_i) + h\frac{dy}{dx}(x_i) + O(h^2) = y(x_i) + f(x_i, y_i) + O(h^2)$$

The local truncation error is $O(h^2)$. It's the difference between the approximate solution after one step and the exact solution at that step.

Let's compute the numerical solution at $x_1 (x_0+h)$:

$$y^*(x_1) = Y_0 + h f(x_0, Y_0)$$

And the exact solution: (Taylor)

$$y(x_0+h) = Y_0 + h\frac{dy}{dx}(x_0) + O(h^2) = Y_0 + h f(x_0, Y_0) + O(h^2)$$

substracting them:

$$y(x_0+h) - y^*(x_1) = \cancel{Y_0} + \cancel{hf(x_0,y_0)} + O(h^2) - (\cancel{Y_0} + \cancel{hf(x_0,y_0)})$$

$$\boxed{LTE = O(h^2)}$$

A method is said to be of order q if the residue is:

$R_i(h) = O(h^{q+1})$ $\rightarrow$ the Euler Method is of order 1

$\boxed{\lim_{h\to 0} LTE = 0}$ $\Rightarrow$ The method is consistent (the error is $\propto$ to h^2. If $h \to 0$, so does the error)

b) In the backward Euler Method a backward approximation of the derivative is considered.

Using Taylor Series:

$$y_i = y_{i+1} - h\frac{dy}{dx}(x_{i+1}) + O(h^2)$$

$$\frac{dy}{dx}(x_{i+1}) = \frac{Y_{i+1} - y_i}{h} + z_i(h)$$ with truncation error $z_i(h) = O(h)$

$\downarrow$

$$y_{i+1} = Y_i + h f(x_{i+1}, Y_{i+1}) + h z_i(h)$$

Neglecting the truncation error :

$$Y_{i+1} = Y_i + h f(x_{i+1}, Y_{i+1}) \Rightarrow \boxed{\begin{array}{l} Y_0 = \alpha \\ Y_{i+1} = Y_i + h f(x_{i+1}, Y_{i+1}) \quad i = 0, \ldots, m-1 \end{array}}$$

An equation (or a system of equations) that might be non-linear has to be solved in order to compute Y_{i+1}. It is an implicit method.

c)

$$\frac{dy}{dx} = -\lambda y$$

Forward Euler method:

$$f(x,y) = -\lambda y$$

$$y_{i+1} = y_i + h(-\lambda y_i) = y_i - h\lambda y_i$$

$$y_{i+1} = G y_i \quad \text{with} \quad G = 1 - \lambda h$$ G is the amplification factor.

The scheme is absolutely stable if $|G| < 1$:

$$|1-\lambda h| < 1 \quad \left\{ \begin{array}{l} 1-\lambda h < 1 \rightarrow -\lambda h < 0 \rightarrow \lambda h > 0 \\ -1+\lambda h < 1 \rightarrow \lambda h < 2 \end{array} \right\} \quad \boxed{0 < \lambda h < 2}$$

Backward Euler Method:

$$f(x,y) = -\lambda y$$

$$y_{i+1} = y_i + h(-\lambda y_{i+1}) = y_{i+1} = y_i - h\lambda y_{i+1} \rightarrow y_i = (1+\lambda h)\, y_{i+1}$$

$$y_{i+1} = G y_i \quad \text{with} \quad G = \frac{1}{1+\lambda h}$$

Let's impose the stability condition:

$$|1+h\lambda| > 1 \quad \left\{ \begin{array}{l} 1+\lambda h > 1 \rightarrow \lambda h > 0 \\ -1-\lambda h > 1 \rightarrow -\lambda h > 2 \rightarrow \lambda h < -2 \end{array} \right\} \quad \boxed{\lambda h > 0 \text{ or } \lambda h < -2}$$

d)

$$\frac{dy}{dx} = -25 y^{3.5} \qquad y(0) = 1 \qquad \text{two steps} \qquad h = 1/10$$

Backward Euler Method:

$$y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})$$

$$y_1 = y_0 + h f(x_{i+1}, y_{i+1}) = 1 + \frac{1}{10}(-25 y_1^{3.5}) \rightarrow y_1 = 1 - 2.5 y_1^{3.5} \rightarrow w(y_1) = +y_1 + 2.5 y_1^{3.5} - 1 = 0$$

We now use 2 Newton iterations. Let's recall the method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad y_1 + 2.5 y_1^{3.5} - 1 = 0$$

$\downarrow$ derivating:

$$1 + 3.5 \cdot 2.5 y_1^{2.5} = 0 \rightarrow 1 + 8.75 y_1^{2.5} \rightarrow w'(y_1) = +8.75 y_1^{2.5} + 1$$

In order to obtain an initial approximation, we are going to neglect y_1 in front of $y_1^{3.5}$:

$$y_1 = \sqrt[3.5]{\frac{1}{2.5}} = 0.7697$$

This way:

$$y_1^{(1)} = y_1^{(0)} - \frac{w_1(y_1^{(0)})}{w_1(y_1^{(0)})} = 0.7697 - \frac{(-0.7698)}{-5.5479} = 0.6309$$

$$y_1^{(2)} = y_1^{(1)} - \frac{w_1(y_1^{(1)})}{w_1(y_1^{(1)})} = 0.6309 - \frac{0.1295}{3.766} = \underline{\underline{0.5965}}$$

we now go back to the backward Euler method:

$$Y_2 = Y_1 + hf(x_2, Y_2) = 0.5925 + \frac{1}{10}(-25 \cdot Y_2^{3.5}) \Rightarrow Y_2 = 0.5925 - 2.5\, Y_2^{3.5}$$

$$w(Y_2) = Y_2 + 2.5 Y_2^{3.5} - 0.5925$$

$$w'(Y_2) = 8.75 Y_2^{3.5} + 1$$

Now we are going to use the Newton method again: (we obtain the first approximation as before)

$$Y_2^{(0)} = \sqrt[3.5]{\frac{0.5965}{2.5}} = 0.662$$

$$Y_2^{(1)} = Y_2^{(0)} - \frac{w_2(Y_2^{(0)})}{w_2'(Y_2^{(0)})} = 0.662 - \frac{0.639}{3.065} = 0.446$$

$$Y_2^{(2)} = Y_2^{(1)} - \frac{w_2(Y_2^{(1)})}{w_2'(Y_2^{(1)})} = 0.446 - \frac{0.0016}{1.517} = \underline{\underline{0.444}}$$

e)

Forward Euler Method:

$$Y_{i+1} = Y_i + hf(x_i, Y_i)$$

$$Y_1 = Y_0 + \frac{1}{10}(-25 \cdot 1^{3.5}) = 1 - 2.5 = -1.5$$

$Y_2 = Y_1 + \frac{1}{10}(-25 \cdot (-1.5)^{3.5}) \Rightarrow$ The result is of complex variable. (it's unstable).

we can check this checking the stability limits:

$$|1 + \lambda h| < 1 \rightarrow |1 - 25h| < 1 \left\{ \begin{array}{ll} 1 - 25h < 1 \rightarrow & h > 0 \\ -1 + 25h < 1 & h < 0.08 \end{array} \right.$$

our h is 0.10, this way, we can see that it is unstable indeed.

f)

using matlab we can prove that:

$\Rightarrow h = 1/10 \rightarrow$ unstable

$\Rightarrow h = 1/15 \rightarrow$ unstable

$\Rightarrow h = 1/30 \rightarrow$ stable $\rightarrow$ this one is the maximum one $\boxed{h_{max} = 1/30}$ which falls in our stability condition $0 < h < 0.08$

$\Rightarrow h = 1/45 \rightarrow$ stable

$\Rightarrow h = 1/90 \rightarrow$ stable