1. The motion of a non-frictional pendulum is governed by the Ordinary Differential Equation (ODE)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where θ is the angular displacement, $L = 1\,m$ is the pendulum length and the gravity acceleration is $g = 9'8\,m/s^2$

The position and velocity at time $t = 1\,s$ are known:

$$\theta(1) = 0'4 \text{ rad} \quad ; \quad \frac{d\theta}{dt}(1) = 0 \text{ rad/s}$$

a) Solve the initial boundary value problem in the interval $(0, 1)$ using a second-order Runge-Kutta method to determine the initial position at $t = 0\,s$, with 2 and 4 time steps.

b) Using the approximation obtained in a), compute an approximation of the relative error in the solution computed with 2 steps

c) Propose a time step h to obtain an approximation with a relative error three orders of magnitude smaller.

a) Reduction of one ODE of order n to a system of n first-order ODEs

$$\left.\begin{matrix} y_1 = \theta \\ y_2 = \theta' \end{matrix}\right] \rightarrow Y = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

$$\left.\begin{matrix} y_1' = \theta' = y_2 \\ y_2' = \theta'' = -\frac{g}{L}\theta = -\frac{g}{L}y_1 \end{matrix}\right] \rightarrow F = \begin{Bmatrix} y_2 \\ -\frac{g}{L}y_1 \end{Bmatrix}$$

Runge-Kutta method:

$$Y_{i+1}^* = Y_i + h f(t_i, Y_i)$$

$$Y_{i+1} = Y_i + \frac{h}{2}\left[f(t_i, Y_i) + f(t_{i+1}, Y_{i+1}^*)\right]$$

2 time steps: $\rightarrow h = \frac{0-1}{2} = -\frac{1}{2}$

(i=0) $Y_{0+1}^* = \begin{Bmatrix} 0'4 \\ 0 \end{Bmatrix} - \frac{1}{2}\begin{Bmatrix} 0 \\ -\frac{98}{25} \end{Bmatrix} = \begin{Bmatrix} 0'4 \\ 49/25 \end{Bmatrix}$

$Y_{0+1} = \begin{Bmatrix} 0'4 \\ 0 \end{Bmatrix} - \frac{1}{4}\left[\begin{Bmatrix} 0 \\ -98/25 \end{Bmatrix} + \begin{Bmatrix} 49/25 \\ -98/25 \end{Bmatrix}\right] = \begin{Bmatrix} 0'4 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 49/100 \\ -49/25 \end{Bmatrix} = \begin{Bmatrix} -0'09 \\ 49/25 \end{Bmatrix} = Y_1$

(i=1) $Y_{1+1}^* = \begin{Bmatrix} -0'09 \\ 49/25 \end{Bmatrix} - \frac{1}{2}\begin{Bmatrix} 49/25 \\ 0'882 \end{Bmatrix} = \begin{Bmatrix} -1'07 \\ 1'519 \end{Bmatrix}$

$Y_{1+1} = \begin{Bmatrix} -0'09 \\ 49/25 \end{Bmatrix} - \frac{1}{4}\left[\begin{Bmatrix} 49/25 \\ 0'882 \end{Bmatrix} + \begin{Bmatrix} 1'519 \\ 10'486 \end{Bmatrix}\right] = \begin{Bmatrix} -0'95975 \\ -0'882 \end{Bmatrix} = Y_2$

$\boxed{\theta(0) = -0'95975}$ $\quad \frac{d\theta}{dt}(0) = -0'882$

Full digitalitzable amb l'app de Notebloc®

Scanned by CamScanner

4 time steps: $\rightarrow h = -\frac{1}{4}$

(repeating the same steps as in the case of 2 time steps)

$i=0$ $\quad Y_{0+1}^{*} = \begin{Bmatrix} 0'4 \\ 198/100 \end{Bmatrix}$ $\qquad$ $i=1$ $\quad Y_{1+1}^{*} = \begin{Bmatrix} 0'0325 \\ 1'6599 \end{Bmatrix}$

$Y_{0+1} = \begin{Bmatrix} 0'2775 \\ 0'98 \end{Bmatrix} = Y_1$ $\qquad$ $Y_{1+1} = \begin{Bmatrix} -0'0525 \\ 1'3598 \end{Bmatrix} = Y_2$

$i=2$ $\quad Y_{2+1}^{*} = \begin{Bmatrix} -0'3924 \\ 1'2312 \end{Bmatrix}$ $\qquad$ $i=3$ $\quad Y_{3+1}^{*} = \begin{Bmatrix} -0'58 \\ -0'1073 \end{Bmatrix}$

$Y_{2+1} = \begin{Bmatrix} -0'3763 \\ 0'8147 \end{Bmatrix} = Y_3$ $\qquad$ $Y_{3+1} = \begin{Bmatrix} -0'4648 \\ -0'3568 \end{Bmatrix} = Y_4$

$\boxed{\theta(0) = -0'4648}$ $\qquad \frac{d\theta}{dt} = -0'3568$

b) Taking into account that the exact value ($\simeq$ 1000 time steps) is

$$Y = \begin{Bmatrix} -0'4 \\ 0'0139 \end{Bmatrix}$$

the error is:

$$\epsilon = \left| \frac{Y_{exact} - Y_{2\,time}}{Y_{exact}} \times 100 \right| = 139'94\ \%$$

c) $h^{*} = \left(\frac{tol}{\epsilon_h}\right)^{1/(p+1)} \cdot h = \left(\frac{10^{-3}\ \cancel{\epsilon_h}}{\cancel{\epsilon_h}}\right)^{1/2} \cdot (-0'5) = -0'016$

Scanned by CamScanner

2. Consider the initial value problem

$$\frac{dy}{dx} = y - x^2 + 1 \qquad x \in (0,1)$$

$$y(0) = 1$$

a) Solve the initial value problem using the Euler method with step $h = 0'25$

b) Compute the solution using the Heun method with a step h such that the computational cost is equivalent to the computational cost in a)

a) Euler method

$$Y_0 = \alpha$$

$$Y_{i+1} = Y_i + h\, f(x_i, Y_i) \quad i = 0, \ldots, m-1$$

i = 0 1 2 3 4

x = 0 0'25 0'5 0'75 1

i=0 $x_0 = 0 \qquad y_0 = 1$

$$y_{0+1} = y_0 + h f(x_0, y_0) = 1 + 0'25\,(1 - 0^2 + 1) = 1'5 = Y_1$$

i=1 $Y_{1+1} = 1'5 + 0'25\,(1'5 - 0'25^2 + 1) = 2'109375 = Y_2$

i=2 $Y_{2+1} = 2'109375 + 0'25\,(2'109375 - 0'5^2 + 1) = 2'82421875 = Y_3$

i=3 $Y_{3+1} = 2'82421875 + 0'25\,(2'82421875 - 0'75^2 + 1) = 3'639648438 = Y_4$

$$\boxed{y(1) = 3'639648438}$$

b) To have the same computational cost: $h = 0'5$

Heun method: $\begin{cases} Y_{i+1}^{*} = Y_i + h f(x_i, Y_i) \\ Y_{i+1} = Y_i + \frac{h}{2}\left[f(x_i, Y_i) + f(x_i, Y_{i+1}^{*})\right] \end{cases}$

i = 0 1 2

x 0 0'5 1

i=0 $Y_{0+1}^{*} = 2$

$Y_{0+1} = 2'1875 = Y_1$

i=1 $Y_{1+1}^{*} = 3'6563$

$Y_{1+1} = 3'8359 = Y_2$

$$\boxed{y(1) = 3'8359}$$

Full digitalitzable amb l'app de Notebloc®

Scanned by CamScanner

3. The ordinary differential equation

$$\frac{dy}{dx} = f(x,y)$$

is defined over the domain $(0,1)$, and is to be solved numerically subjected to the initial condition $y(0)=1$, where $y(x)$ is the exact solution. the forward Euler method for integrating the above differential equation is written as

$$Y_{i+1} = Y_i + h f(x_i, Y_i)$$

where Y_i denotes the discrete solution at node i, with position x_i of a uniform grid of nodes of constant grid interval size h and $x_{i+1} = x_i + h$.

a) Using a Taylor series expansion, deduce the leading truncation error of the scheme. Is the method consistent? Explain your answer.

b) State the backward Euler method for integrating the above differential equation where $f(x,y)$ is a general non-linear function of x and y

c) Deduce the stability limits of the respective forward Euler method and backward Euler method for the model equation $dy/dx = -\lambda y$ where λ is a positive real constant.

d) Use the backward Euler method to compute the numerical solution of the ordinary differential equation

$$\frac{dy}{dx} = -25y^{3.5}$$

with initial conditions $y(0)=1$, by hand for two steps with grid interval size $h = 1/10$ (use 2 Newton iterations per step for this calculation)

e) Use the forward Euler method to compute the numerical solution of the above ordinary differential equation with same initial condition by hand for two steps with grid interval size $h = 1/10$

f) The analitical solution is

$$y(x) = \left(\frac{125x + 2}{2}\right)^{-2/5}$$

Using Matlab codes, indicate the maximun stable interval size possible for forward Euler method from the following: $h = 1/10$, $h = 1/15$, $h = 1/30$, $h = 1/45$, $h = 1/90$. How does your choice compare with the stability condition.

Scanned by CamScanner

a) $y_{i+1} = y_i + h \frac{dy}{dx}(x_i) + \frac{h^2}{2} \frac{d^2y}{dx^2}(x_i) + O(h^3)$

$y_{i-1} = y_i - h \frac{dy}{dx}(x_i) + \frac{h^2}{2} \frac{d^2y}{dx^2}(x_i) + O(h^3)$

substracting

$$y_{i+1} - y_{i-1} = 2h \frac{dy}{dx}(x_i) + O(h^3)$$

$$\frac{dy}{dx}(x_i) = \frac{y_{i+1} - y_{i-1}}{2h} - \boxed{\tau_i(h)}$$

$\tau_i(h) = O(h^2)$ truncation error

b) $y = y_{i+1} - h \frac{dy}{dx}(x_{i+1}) + O(h^2)$

$$\frac{dy}{dx}(x_{i+1}) = \frac{y_{i+1} - y_i}{h} + \boxed{\tau_i(h)}$$

$\tau_i(h) = O(h)$ truncation error

Replacing the ODE $\to y_{i+1} = y_i + h f(x_{i+1}, y_{i+1}) + h\tau_i(h)$

Neglecting the truncation error the numerical scheme for the backward Euler is obtained

$$y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})$$

c) when $Re(\lambda) < 0$ the analitical solution goes to zero (when x goes to infinitive.

For fixed values of λ and h, the scheme is said to be absolutely stable if the numerical solution goes to zero.

Backward Euler methot for $f(x,y) = \lambda y$

That is $y_{i+1} = y_i + h\lambda y_{i+1} \longrightarrow (1 - h\lambda) y_{i+1} = y_i$

$y_{i+1} = G y_i$ with $G = \frac{1}{1 - h\lambda}$

stability condition: $|1 - \lambda h| > 1$

stabitity condition for real λ: $\underbrace{\lambda h < 0 \text{ or } \lambda h > 2}_{\text{unconditionally stable for } \lambda < 0}$

Full digitalitzable amb l'app de Notebloc®

Scanned by CamScanner

d) $y_0 = 1$

$i=0$ $y_{0+1} = y_0 + \frac{1}{10}(-25\, y_1^{3'5}) = y_1$

$2'5\, y_1^{3'5} + y_1 - 1 = 0$

$\Downarrow$

Newton $\boxed{x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}}$

$x^0 = 1$

$$x^1 = x^0 - \frac{2'5\sqrt{x^{0\,7}} + x^0 - 1}{\frac{8'75\sqrt{x^{0\,7}}}{x^0} + 1} = 0'8462$$

$$x^2 = x^1 - \frac{2'5\sqrt{x^{1\,7}} + x^1 - 1}{\frac{8'75\sqrt{x^{1\,7}}}{x^1} + 1} = 0'7880 \rightarrow y_1 = 0'7880$$

$i=1$ $y_{1+1} = y_1 + \frac{1}{10}(-25 \cdot y_2^{3'5}) = y_2$

$2'5\sqrt{y_2^7} + y_2 - 0'7880 = 0$

$\Downarrow$

Newton

$x^0 = 0'7880$

$$x^1 = x^0 - \frac{2'5\sqrt{x^{0\,7}} + x^0 - 0'7880}{\frac{8'75\sqrt{x^{0\,7}}}{x^0} + 1} = 0'7368$$

$x^2 = 0'7229 \rightarrow y_2 = 0'7229$

e) $y_1 = y_0 + h f(x_0, y_0)$

$i=0$ $y_{0+1} = 1 + \frac{1}{10}(-25 \cdot 1^{3'5}) = -1'5 = y_1$

$i=1$ $y_{1+1} = -1'5 + \frac{1}{10}(-25(-1'5)^{3'5}) \Rightarrow$ UNSTABLE

NOTEBLOC®

Scanned by CamScanner