1*. In 1225, Leonardo of Pisa (also know as Fibonacci) was requested to solve a collection of mathematical problems in order to justify his fame in the court of Federico II. One of the proposed problems can be formulated as the solution of a third degree polynomial equation:

$$f(x) := x^3 + 2x^2 + 10x - 20 = 0 \qquad (1)$$

Note that the solution of cubic equations was a extremly difficult problem in the 13th century. Here iterative methods are considered for the solution of the equation (1).

Compute the unique real root of (1) with 4 iterations of Newton's method with the initial approximation $x^0 = \sqrt[3]{20}$ (which is obtained neglecting the monomials with x and x^2 in front of the monomials with x^3). Plot the convergence graphic. Does Newton's methot behave as expected?

Newton's method approximate functions by its tangent line (first-order Taylor expansion) and impose that the next approximation be the solution of the linear equation.

$$\boxed{x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}}$$

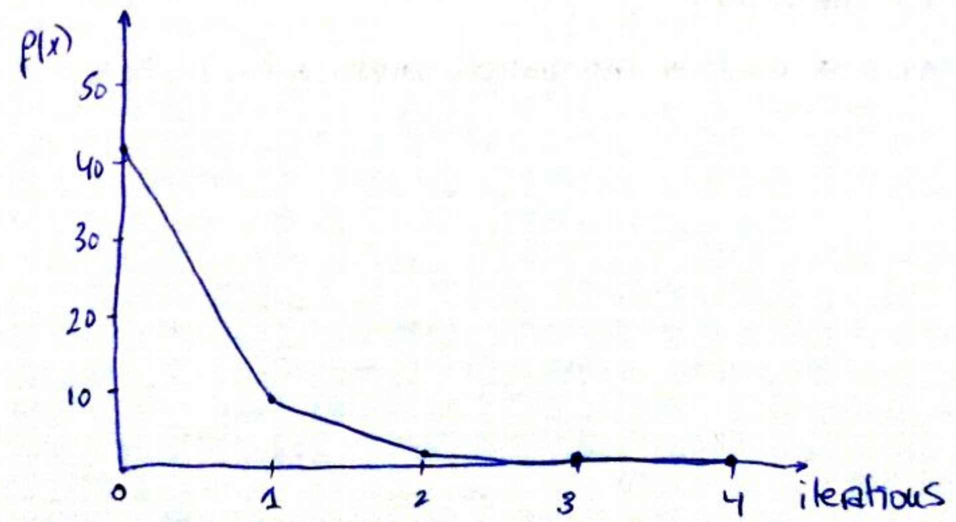
$$x^0 = \sqrt[3]{20}$$

$$x^1 = x^0 - \frac{x^{0^3} + 2x^2 + 10x - 20}{3x^2 + 4x + 10} = \sqrt[3]{20} - \frac{20 + 2\sqrt[3]{20^2} + 10\sqrt[3]{20} - 20}{3\sqrt[3]{20^2} + 4\sqrt[3]{20} + 10} = 1'7396$$

$$x^2 = x^1 - \frac{x^3 + 2x^2 + 10x - 20}{3x^2 + 4x + 10} = 1'4049$$

$$x^3 = x^2 - \frac{x^3 + 2x^2 + 10x - 20}{3x^2 + 4x + 10} = 1'3692$$

$$x^4 = x^3 - \frac{x^3 + 2x^2 + 10x - 20}{3x^2 + 4x + 10} = 1'3688$$



5*. We are interested in the definition of third-order numerical quadratures in interval $(0,1)$

a) Determine the minimum number of integration points, and specify the integration points and weights.

c) It is possible to obtain a third-order quadrature with the following four integration points: $x_0 = 1/4$, $x_1 = 1/2$, $x_2 = 3/4$, $x_3 = 1$? If it is possible, compute the corresponding weights; otherwise, justify why not.

a) $3 = 2n+1 \rightarrow n = 1 \Rightarrow$ integration points $\{z_0, z_1\}$ (Gauss

$$\left.\begin{aligned}\int_0^1 1\,dx &= 1 = w_0 + w_1\\ \int_0^1 x\,dx &= \frac{1}{2} = w_0 z_0 + w_1 z_1\\ \int_0^1 x^2 dx &= \frac{1}{3} = w_0 z_0^2 + w_1 z_1^2\\ \int_0^1 x^3 dx &= \frac{1}{4} = w_0 z_0^3 + w_1 z_1^3\end{aligned}\right\} \quad \begin{aligned} w_0 &= \frac{1}{2} = w_1\\ z_0 &= \frac{1}{6}(3-\sqrt{3})\\ z_1 &= \frac{1}{6}(3+\sqrt{3})\end{aligned}$$

b)
$$\left.\begin{aligned} w_0 + w_1 + w_2 + w_3 &= 1\\ w_0 \cdot \frac{1}{4} + w_1 \cdot \frac{1}{2} + w_2 \frac{3}{4} + w_3 &= \frac{1}{2}\\ w_0 \cdot \frac{1}{16} + w_1 \cdot \frac{1}{4} + w_2 \frac{9}{16} + w_3 &= \frac{1}{3}\\ w_0 \frac{1}{64} + w_1 \frac{1}{8} + w_2 \frac{27}{64} + w_3 &= \frac{1}{4}\end{aligned}\right\} \quad \begin{aligned} w_0 &= \frac{2}{3}\\ w_1 &= -\frac{1}{3}\\ w_2 &= \frac{2}{3}\\ w_3 &= 0\end{aligned}$$

As we can see it is possible to obtain a third-order quadrature with these points but one of them is useless as one of the weights gives zero.

To check this we can apply Newton-Cotes since we know the values of the points.

$3 = n+1 \rightarrow n = 2 \rightarrow$ integration points $\{z_0, z_1, z_2\}$

6*. a) If $n+1$ points Gaussian quadrature is used for numerical integration state the order of the polynomial that is integrated exactly

b) If $n=2$ is selected for Gaussian quadrature, which (if any) of the following integrals will be integrated exactly?

i) $\int_0^1 \sin x \, dx$ ii) $\int_0^1 x^3 dx$ iii) $\int_0^1 x^4 dx$ iv) $\int_0^1 x^{5.5} dx$

a) $n+1 \rightarrow n = n^* \rightarrow$ order $2n^*+1$

b) ii) and iii) will be integrated exactly as they are of minor order

In the case of i) and iv) the solution won't be integrated exacty.

7*. Compute $\int_0^1 12x\,dx$, $\int_0^1 (5x^3+2x)\,dx$ by hand calculation using

a) Trapezoidal rule over 2 uniform integrals

b) Simpson's rule over 2 uniform integrals

Compute the error of both approximations. Are the methods behaving as expected?

a) Trapezoidal rule $\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2f(a+h) + 2f(a+2h) + \dots + f(b)\right]$

$\rightarrow \int_0^1 12x\,dx = \frac{1}{4}\left(12\cdot 0 + 2\cdot 12\left(\frac{1}{2}\right) + 12\cdot 1\right) = 6 \rightarrow \text{Error} = 0$

$\rightarrow \int_0^1 (5x^3+2x)dx = \frac{1}{4}\left[(5\cdot 0^3 + 2\cdot 0) + 2\cdot(5\cdot 0'5^3 + 2\cdot 0'5) + (5\cdot 1^3 + 2\cdot 1)\right] =$

$= 2'5625 \rightarrow \text{Error} = 0'3125$

b) Simpson's rule $\rightarrow \int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 4f(a+h) + f(b)\right]$

$\rightarrow \int_0^1 12x\,dx = \frac{1}{6}\left(0 + 4\cdot 12 + 12\cdot 1\right) = 6 \rightarrow \text{Error} = 0$

$\rightarrow \int_0^1 (5x^3+2x)dx = \frac{1}{6}\left(0 + \frac{13}{2} + 7\right) = 2'25 \rightarrow \text{Error} = 0$

10*. Perform the numerical integration of

$$\int_0^1\int_0^1 (9x^3+8x^2)(y^3+y)\,dx\,dy$$

using Simpson's rule in each direction. Is the approximation behaving as expected?

$$x: \int_0^1 (9x^3+8x^2)\,dx = \frac{1}{6}\left(0+4\left(9\tfrac{1}{8}+8\tfrac{1}{4}\right)+(9\cdot 1+8\cdot 1)\right) =$$

$$= 4'9166$$

$$y: \int_0^1 (y^3+y)\,dy = \frac{1}{6}\left(0+4\left(\tfrac{1}{8}+\tfrac{1}{2}\right)+(1^3+1)\right) = 0'75$$

$$\int_0^1\int_0^1 (9x^3+8x^2)(y^3+y)\,dx\,dy = 4'9166\cdot 0'75 = \underline{3'6875}$$