Solution 1

Given ODE

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

$\theta \rightarrow$ angular displacement ; $L = 1m$; $g = 9.81\ m/s^2$

Position and velocity at $t=1$

$\theta(1) = 0.4$ rad ; $\frac{d\theta}{dt}(1) = 0$ rad/s

(a) Solve IVP in interval $(0,1)$ using second order RK method with 2 time steps $h = \frac{1-0}{2} = 0.5$

We can reduce the 2nd order ODE to two first order ODE

So, Let $\theta_1 = \theta$ and $\frac{d\theta_1}{dt} = \theta_2 = \frac{d\theta}{dt} = \theta_1'$

$\frac{d\theta_1}{dt} = \theta_2$

Substituting in ODE

$$\frac{d\theta_2}{dt} + g\theta_1 = 0 \Rightarrow \frac{d\theta_2}{dt} = -g\theta_1$$

We know that $\theta_1(1) = 0.4$ and $\theta_2(1) = 0$

$$\frac{d\theta}{dt} = f(t,\theta) = \begin{bmatrix} \theta_2 \\ -g\theta_1 \end{bmatrix}$$

Using second order RK method

$\theta_{i+1} = \theta_i + \frac{h}{2}[K_1 + K_2]$ where $K_1 = f(t_i, \theta_i)$

$K_2 = f(t_i + h, \theta_i + hK_1)$

$K_1 = f(t_i, \theta_i)$

$$K_1^{(1)} = \begin{bmatrix} 0 \\ -g(0.4) \end{bmatrix} \quad \text{i.e at } t=1$$

$$\theta^{(0)} = \begin{bmatrix} \theta_1^{(0)} \\ \theta_2^{(0)} \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$$

$$\theta^{(1)} = \theta^{(0)} - \frac{h}{2}[K_1^{(1)} + K_2^{(1)}]$$

$$K_2^{(1)} = f(t_i - h, \theta_i - hK_1) = f\left(t_i - h, \theta^{(0)} - 0.5\begin{bmatrix} 0 \\ -g(0.4) \end{bmatrix}\right)$$

$$= f\left(t_i - h, \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - 0.5\begin{bmatrix} 0 \\ -g(0.4) \end{bmatrix}\right)$$

$$K_2^{(1)} = f\left(t_1 - h, \begin{bmatrix} 0.4 \\ g(0.4)(0.5) \end{bmatrix}\right)$$

$$K_2^{(1)} = \begin{bmatrix} g(0.4 \times 0.5) \\ -g(0.4) \end{bmatrix} = \begin{bmatrix} g(0.2) \\ -g(0.4) \end{bmatrix}$$

$$\theta^{(1)} = \theta^{(0)} - \frac{h}{2}[K_1 + K_2]$$

$$= \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - \frac{0.5}{2}\left\{\begin{bmatrix} 0 \\ -g(0.4) \end{bmatrix} + \begin{bmatrix} g(0.2) \\ -g(0.4) \end{bmatrix}\right\}$$

$$\theta^{(1)} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - \frac{0.5}{2}\begin{bmatrix} 1.96 \\ -7.84 \end{bmatrix} = \begin{bmatrix} 0.09 \\ +1.96 \end{bmatrix} = \begin{bmatrix} \theta_1^{(1)} \\ \theta_2^{(1)} \end{bmatrix}$$

$$\theta^{(2)} = \theta^{(1)} - \frac{h}{2}[K_1^{(2)} + K_2^{(2)}]$$

$$K_1^{(2)} = f(t^{(1)}, \theta^{(1)}) = \begin{bmatrix} 1.96 \\ -g(0.09) \end{bmatrix}$$

$$K_2^{(2)} = f(t^{(1)} - h, \theta^{(1)} - h K_1)$$

$$= f\left(t^{(1)} - h, \begin{bmatrix} 0.09 \\ 1.96 \end{bmatrix} - 0.5\begin{bmatrix} 1.96 \\ -0.882 \end{bmatrix}\right)$$

$$K_2^{(2)} = f\left(t^{(1)} - h, \begin{bmatrix} -0.89 \\ 2.401 \end{bmatrix}\right) = \begin{bmatrix} 2.401 \\ g(0.89) \end{bmatrix}$$

$$\theta^{(2)} = \theta^{(1)} - \frac{h}{2}[K_1^{(2)} + K_2^{(2)}]$$

$$= \begin{bmatrix} 0.09 \\ 1.96 \end{bmatrix} - \frac{0.5}{2}\left\{\begin{bmatrix} 1.96 \\ -g(0.09) \end{bmatrix} + \begin{bmatrix} 2.401 \\ g(0.89) \end{bmatrix}\right\}$$

$$\theta^{(2)} = \begin{bmatrix} -1.00025 \\ 0 \end{bmatrix}$$

\# Now solving IVP in interval (0, 1) with 4 time steps

$$h = \frac{1-0}{4} = 0.25$$

As shown before $\frac{d\theta}{dt} = f(t, \theta) = \begin{bmatrix} \theta_2 \\ -g\theta_1 \end{bmatrix}$

$$\theta^{(0)} = \begin{bmatrix} \theta_1(1) \\ \frac{d\theta(1)}{dt} \end{bmatrix} = \begin{bmatrix} \theta_1^{(0)} \\ \theta_2^{(0)} \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$$

As per second order RK method

$$\theta_{i+1} = \theta_i - \frac{h}{2}[K_1 + K_2]$$

$$K_1 = f(t_i, \theta_i) \quad \text{and} \quad K_2 = f(t_i + h, \theta_i + hK_1)$$

$$K_1^{(1)} = f(t^{(0)}, \theta^{(0)}) = \begin{bmatrix} 0 \\ -g(0.4) \end{bmatrix}$$

$$K_2^{(1)} = f(t^{(0)} - h, \theta^{(0)} - hK_1^{(1)})$$

$$= f\left(t^{(0)} - h, \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - 0.25 \begin{bmatrix} 0 \\ -g(0.4) \end{bmatrix}\right)$$

$$= f\left(t^{(0)} - h, \begin{bmatrix} 0.4 \\ g(0.1) \end{bmatrix}\right)$$

$$K_2^{(1)} = \begin{bmatrix} g(0.1) \\ -g(0.4) \end{bmatrix}$$

$$\theta^{(1)} = \theta^{(0)} - \frac{h}{2}\left[K_1^{(1)} + K_2^{(1)}\right] = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - \frac{0.25}{2}\left\{\begin{bmatrix} 0 \\ -g(0.4) \end{bmatrix} + \begin{bmatrix} g(0.1) \\ -g(0.4) \end{bmatrix}\right\}$$

$$\theta^{(1)} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0.1225 \\ -0.98 \end{bmatrix} = \begin{bmatrix} 0.2775 \\ +0.98 \end{bmatrix} \begin{bmatrix} \theta_1^{(1)} \\ \theta_2^{(1)} \end{bmatrix}$$

$$\theta^{(2)} = \theta^{(1)} - \frac{h}{2}\left[K_1^{(2)} + K_2^{(2)}\right]$$

$$K_1^{(2)} = f(t^{(1)}, \theta^{(1)}) = \begin{bmatrix} 0.98 \\ -g(0.2775) \end{bmatrix} = \begin{bmatrix} 0.98 \\ -2.7195 \end{bmatrix}$$

$$K_2^{(2)} = f(t^{(1)} - h, \theta^{(1)} - hK_1^{(2)})$$

$$= f\left(t^{(1)} - h, \begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix} - 0.25 \begin{bmatrix} 0.98 \\ -g(0.2775) \end{bmatrix}\right)$$

$$K_2^{(2)} = f\left(t^{(1)} - h, \begin{bmatrix} 0.0325 \\ 1.6599 \end{bmatrix}\right) = \begin{bmatrix} 1.6599 \\ -g(0.0325) \end{bmatrix}$$

$$\theta^{(2)} = \begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix} - \frac{0.25}{2}\left\{\begin{bmatrix} 0.98 \\ -2.7195 \end{bmatrix} + \begin{bmatrix} 1.6599 \\ 0.3185 \end{bmatrix}\right\}$$

$$\theta^{(2)} = \begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix} - \begin{bmatrix} 0.32998 \\ -0.300125 \end{bmatrix} = \begin{bmatrix} -0.05249 \\ 1.280125 \end{bmatrix}$$

$$\theta^{(3)} = \theta^{(2)} - \frac{h}{2}\left[K_1^{(3)} + K_2^{(3)}\right]$$

$$K_1^{(3)} = f(t^{(2)}, \theta^{(2)}) = \begin{bmatrix} 1.280125 \\ +g(0.05249) \end{bmatrix} = \begin{bmatrix} 1.280125 \\ 0.514402 \end{bmatrix}$$

$$K_2^{(3)} = f\left(t^{(2)} - h,\ \theta^{(2)} - h K_1^{(3)}\right)$$

$$= f\left(t^{(2)} - h,\ \begin{bmatrix} -0.05249 \\ 1.280125 \end{bmatrix} - 0.25\begin{bmatrix} 1.280125 \\ 0.514402 \end{bmatrix}\right)$$

$$= f\left(t^{(2)} - h,\ \begin{bmatrix} -0.3725 \\ 1.15152 \end{bmatrix}\right)$$

$$K_2^{(3)} = \begin{bmatrix} 1.15152 \\ +g(0.3725) \end{bmatrix} = \begin{bmatrix} 1.15152 \\ 3.6505 \end{bmatrix}$$

$$\theta^{(3)} = \theta^{(2)} - \frac{h}{2}\left[K_1^{(3)} + K_2^{(3)}\right]$$

$$= \begin{bmatrix} -0.05249 \\ 1.280125 \end{bmatrix} - \frac{0.25}{2}\left\{\begin{bmatrix} 1.280125 \\ 0.514402 \end{bmatrix} + \begin{bmatrix} 1.15152 \\ 3.6505 \end{bmatrix}\right\}$$

$$\theta^{(3)} = \begin{bmatrix} -0.05249 \\ 1.280125 \end{bmatrix} - \begin{bmatrix} 0.30396 \\ 0.520625 \end{bmatrix} = \begin{bmatrix} -0.35645 \\ 0.7595 \end{bmatrix}$$

$$\theta^{(4)} = \theta^{(3)} - \frac{h}{2}\left[K_1^{(4)} + K_2^{(4)}\right]$$

$$K_1^{(4)} = f(t^{(3)}, \theta^{(3)}) = f\left(t^{(3)},\ \begin{bmatrix} -0.35645 \\ 0.7595 \end{bmatrix}\right)$$

$$K_1^{(4)} = \begin{bmatrix} 0.7595 \\ g(0.35645) \end{bmatrix} = \begin{bmatrix} 0.7595 \\ 3.49321 \end{bmatrix}$$

$$K_2^{(4)} = f\left(t^{(4)} - h,\ \theta^{(3)} - h K_1^{(4)}\right)$$

$$= f\left(t^{(4)} - h,\ \begin{bmatrix} -0.35645 \\ 0.7595 \end{bmatrix} - 0.25\begin{bmatrix} 0.7595 \\ 3.49321 \end{bmatrix}\right)$$

$$= f\left(t^{(4)} - h,\ \begin{bmatrix} -0.54633 \\ -0.1138025 \end{bmatrix}\right)$$

$$K_2^{(4)} = \begin{bmatrix} -0.1138025 \\ g(0.54633) \end{bmatrix} = \begin{bmatrix} -0.1138025 \\ 5.353985 \end{bmatrix}$$

$$\theta^{(4)} = \begin{bmatrix} -0.35645 \\ 0.7595 \end{bmatrix} - \frac{0.25}{2}\left\{\begin{bmatrix} 0.7595 \\ 3.49321 \end{bmatrix} + \begin{bmatrix} -0.1138025 \\ 5.353985 \end{bmatrix}\right\}$$

$$\theta^{(4)} = \begin{bmatrix} -0.4372 \\ -0.3464 \end{bmatrix}$$

(b) θ from 2 step $= -1.00025$
θ from 4 step $= -0.4372$

$$\text{relative error} = \frac{\theta \text{ of } 2 - \theta \text{ of } 4}{\theta \text{ of } 4}$$

$$= \frac{-1.00025 + 0.4372}{-0.4372}$$

$$= 1.287$$

(c) using formula

$$h^* = \left(\frac{tol}{E_n}\right)^{1/(p+1)} \quad \text{where } \frac{tol}{E_n} = 10^{-3}$$

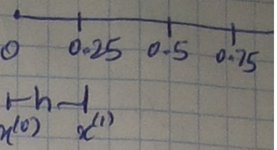
$$h = 0.5, \quad p = 2$$

$$h^* = (10^{-3})^{1/3}(0.5) = 0.05$$

$$\text{time step} = \frac{1-0}{0.05} = 20 \text{ steps}$$

Solution 2 Given IVP $\frac{dy}{dx} = y - x^2 + 1$ $x \in (0,1)$ and $y(0) = 1$

(a) Solve IVP with Euler method and step $h = 0.25$



Euler Method : $Y_{i+1} = Y_i + h\, f(x_i, Y_i)$

Given ODE $\frac{dy}{dx} = f(x,y) = y - x^2 + 1$

$y^{(0)} = y(0) = 1$ i.e. at $x = 0 = x^{(0)}$

$x^0 = 0$

$y^{(1)} = y^{(0)} + 0.25\, f(x^{(0)}, y^{(0)})$

$= 1 + 0.25(1+1) = 1.5$

$y^{(2)} = y^{(1)} + 0.25\, f(x^{(1)}, y^{(1)})$ $x^{(1)} = x^0 + h$

$= 1.5 + 0.25[1.5 - (0.25)^2 + 1]$ $= 0 + 0.25 = 0.25$

$= 2.1094$

$y^{(3)} = y^{(2)} + 0.25\, f(x^{(2)}, y^{(2)})$ $x^{(2)} = x^{(1)} + h$

$= 2.1094 + 0.25[2.1094 - (0.5)^2 + 1]$ $= 0.25 + 0.25 = 0.5$

$= 2.8243$

$y^{(4)} = y^{(3)} + 0.25\, f(x^{(3)}, y^{(3)})$ $x^{(3)} = x^{(2)} + h$

$= 2.8243 + 0.25[2.8243 - (0.75)^2 + 1]$ $= 0.5 + 0.25$

$= 3.63975$ $= 0.75$

$x^{(4)} = x^{(3)} + h$

$= 0.75 + 0.25$

$= 1$

Using Heun Method

$y_{i+1} = y_i + \frac{h}{2}[K_1 + K_2]$ and $K_1 = f(x_i, y_i)$

$K_2 = f(x_i + h, y_i + hK_1)$

$y^{(1)} = y^{(0)} + \frac{h}{2}[K_1^{(1)} + K_2^{(1)}]$

$K_1^{(1)} = f(x^{(0)}, y^{(0)}) = 1 - 0 + 1 = 2$

$K_2^{(1)} = f(x^{(0)} + h, y^{(0)} + h K_1^{(1)}) = f(0.25, \{1 + 0.25(2)\})$

$= f(0.25, 1.5) = 1.5 - (0.25)^2 + 1 = 2.4375$

$y^{(1)} = y^{(0)} + \frac{h}{2}[K_1^{(1)} + K_2^{(1)}] = 1 + \frac{0.25}{2}[2 + 2.4375]$

$= 1.5547$

~~2D uniform flow~~

$$y^{(2)} = y^{(1)} + \frac{h}{2}\left[K_1^{(2)} + K_2^{(2)}\right]$$

$$K_1^{(2)} = f(x^{(1)}, y^{(1)}) = f(0.25, 1.5547) = 1.5547 - (0.25)^2 + 1$$

$$K_1^{(2)} = 2.4922$$

$$K_2^{(2)} = f(x^{(1)}+h, y^{(1)} + hK_1^{(2)}) = f\left(0.5, \{1.5547 + 0.25(2.4922)\}\right)$$

$$= f(0.5, 2.1778) = 2.1778 - (0.5)^2 + 1$$

$$= 2.9278$$

$$y^{(2)} = y^{(1)} + \frac{h}{2}\left[K_1^{(2)} + K_2^{(2)}\right]$$

$$= 1.5547 + \frac{0.25}{2}\left[2.4922 + 2.9278\right]$$

$$= 2.2322$$

$$y^{(3)} = y^{(2)} + \frac{h}{2}\left[K_1^{(3)} + K_2^{(3)}\right]$$

$$K_1^{(3)} = f(x^{(2)}, y^{(2)}) = f(0.5, 2.2322)$$

$$= 2.2322 - (0.5)^2 + 1 = 2.9822$$

$$K_2^{(3)} = f(x^{(2)}+h, y^{(2)} + hK_1^{(3)}) = f\left(0.75, \{2.2322 + 0.25(2.9822)\}\right)$$

$$= f(0.75, 2.9778) = 2.9778 - (0.75)^2 + 1$$

$$= 3.4153$$

$$y^{(3)} = 2.2322 + \frac{0.25}{2}\left[2.9822 + 3.4153\right]$$

$$= 3.03189$$

$$y^{(4)} = y^{(3)} + \frac{h}{2}\left[K_1^{(4)} + K_2^{(4)}\right]$$

$$K_1^{(4)} = f(x^{(3)}, y^{(3)}) = f(0.75, 3.03189)$$

$$= 3.03189 - (0.75)^2 + 1 = 3.4694$$

$$K_2^{(4)} = f(x^{(3)}+h, y^{(3)} + hK_1^{(4)}) = f\left(1, \{3.03189 + 0.25(3.4694)\}\right)$$

$$= f(1, 3.899) = 3.899 - (1)^2 + 1 = 3.899$$

Solution 3 (a) using Taylor series

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + \text{---} \text{---}$$

$$\frac{dy}{dx} = \frac{y_{i+1} - y_i}{h} - \frac{h}{2!}\frac{d^2y}{dx^2}$$

$$\frac{dy}{dx} = \frac{y_{i+1} - y_i}{h} - \tau(h)$$

where $\tau(h) = O(h)$ is truncation error

For method to be consistent

$$\max \tau_i(h) \to 0 \quad \text{when } h \to 0$$

so, when $h \to 0$ the $\max \tau(h) = 0$ hence it is consistent

(b) Backward Euler method

$$y_i = y_{i+1} - h\frac{dy}{dx} + O(h^2)$$

$$\frac{dy}{dx} = \frac{y_{i+1} - y_i}{h} + \tau(h)$$

$$y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1}) + h\,\tau_i(h)$$

neglecting error terms

$$y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1})$$

(c) Stability limit for forward Euler

$$\frac{dy}{dx} = -\lambda y$$

$$y_{i+1} = y_i - h\lambda y_i$$

$$y_{i+1} = G y_i \quad \text{with } G = 1 - h\lambda$$

The scheme is stable in $|G| < 1$

$$|1 - h\lambda| < 1$$

$$h\lambda > 0 \quad \text{and} \quad h\lambda < 2$$

$$\boxed{2 > h\lambda > 0}$$

Stability limit for Backward Euler Method

$$\frac{dy}{dx} = -\lambda y$$

$$y_{i+1} = y_i - h\lambda y_{i+1}$$

$$(1+h\lambda) y_{i+1} = y_i$$

$$y_{i+1} = G y_i \quad \text{with} \quad G = \frac{1}{1+h\lambda}$$

It is stable if $|1+h\lambda| > 1$

$h\lambda > 0$ $h\lambda < -2$ $\Rightarrow$ conditional stability

(d) $\frac{dy}{dx} = -25y^{(3.5)}$ $\quad y(0) = 1$, $h = 1/10$

when $i = 0$ then $y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})$

$$y_1 = y_0 + h f(x_1, y_1)$$

$$y_1 = 1 + \frac{1}{10}\left[-25 y_1^{3.5}\right]$$

$$\frac{25}{10} y_1^{(3.5)} + y_1 - 1 = 0$$

$$f(y) = \frac{25}{10} y_1^{3.5} + y_1 - 1 = 0$$

$$f'(y) = 8.75 y_1^{2.5} + 1$$

using Newton method with $(y_1)_0 = 1$

$$(y_1)_1 = (y_1)_0 - \frac{f(y)}{f'(y)} = 1 - \frac{2.5}{9.75} = 0.74358$$

$$(y_1)_2 = (y_1)_2 - \frac{f(y)}{f'(y)} = 0.74358 - \frac{0.629892}{5.171836}$$

$$= 0.62178$$

For $i = 2$

$$y_2 = y_1 + h f(x_2, y_2)$$

$$y_2 = 0.62178 + \frac{1}{10} f(25 y_2^{3.5}) \Rightarrow 2.5 y_2^{3.5} + y_2 - 0.62178 = 0$$

$$f'(y_2) = 8.75 y_2^{2.5} + 1$$

using Newton method with initial value $(y_2)_0 = 0.62178$

$$(y_2)_1 = (y_1)_0 - \frac{f(y_2)}{f'(y_2)} = 0.62178 - \frac{0.47388}{3.66747}$$

$$= 0.49257$$

$$(y_2)_2 = (y_2)_1 - \frac{f(y_2)}{f'(y_2)} = 0.49257 - \frac{0.0804801}{2.489}$$

$$= 0.4602$$

(e) $\frac{dy}{dx} = -25y^{3.5}$

$y(0) = 1$, $x_0 = 0$, $y_0 = 1$, $h = \frac{1}{10}$

$y_{i+1} = y_i + h\,f(x_i, y_i)$

for $i = 0 \Rightarrow y_1 = y_0 + h\,f(x_0, y_0)$

$y_1 = 1 + \frac{1}{10} f(0,1) = 1 - 2.5 = -1.5$

for $i = 1 \Rightarrow y_2 = y_1 + h\,f(x_1, y_1)$

$= -1.5 + \frac{1}{10} f\left(\frac{1}{10}, -1.5\right)$

$= -1.5 + \frac{1}{10}\left(-25\,(1.5)^{3.5}\right)$

Since the value is increasing so solution is unstable.