Homework 2

Borja Elguerabal

① $f(x) = x^3 + 2x^2 + 10x - 20 = 0$

$x_0 = \sqrt[3]{20}$

4 iteration's of Newton's method.

$f'(x) = 3x^2 + 4x + 10$

Newton - Raphson method: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

1st iteration

$f(x_0) = f(\sqrt[3]{20}) = 41,8803$

$f'(\sqrt[3]{20}) = 42,9619$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1'7396$

2nd iteration

$f(x_1) = f(1'7396) = 8'7126$

$f'(x_1) = f'(1'7396) = 26,0369$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1'4050$

3rd iteration

$f(x_2) = f(1'4050) = 0'7708$

$f'(x_2) = f'(1'4050) = 21,5417$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1'3692$

4th iteration

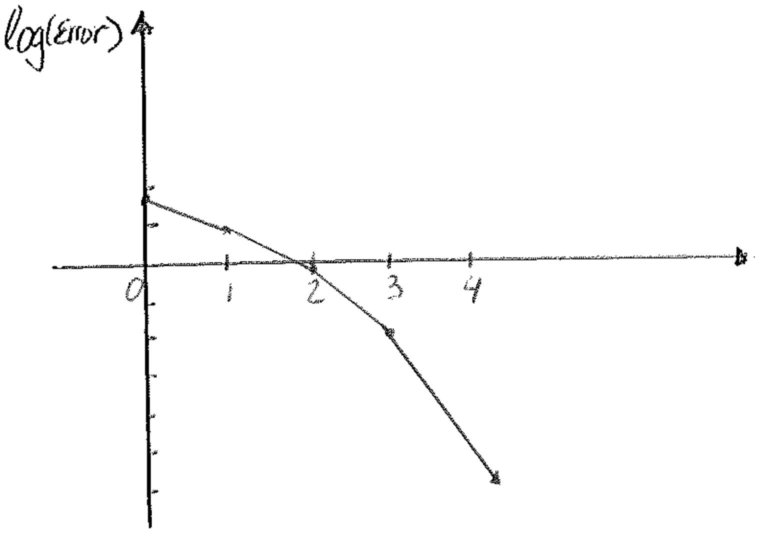
$f(x_3) = f(1'3692) = 0'0079$

$f'(x_3) = f'(1'3692) = 21,1007$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1,3688$

$f(x_4) = f(1,3688) = 8'5872 \cdot 10^{-7}$

Iteration	Error	log(Error)
1	8'7126	0'940147
2	0'7708	-0'1130
3	0'0079	-2,1023
4	$8'58872 \cdot 10^{-7}$	-6,066



It behaves as expected because it has a quadratic convergence near the root.

(5) $\int_0^1 f(x)\,dx = \sum_i^n w_i f(z_i)$ third-order quadrature.

a) The error in a Gauss Quadrature takes the form:

$$E_n = \Omega_n f^{(2n+2)}(\mu) \qquad n+1: \text{Number of points}$$

In order to integrate exactly a third order polynomial (Third order quadrature) two points are required $(n=1)$. So, the error takes the form.

$$E_1 = \Omega_1 f^{(4)}(\mu)$$

$$\int_0^1 f(x)dx = \underbrace{\int_a^b F(t)\,dt = \frac{b-a}{2}}_{\text{Variable change}}\int_{-1}^1 f(z)\,dz \simeq \frac{b-a}{2}\left(w_0 f(z_0) + w_1 f(z_1)\right)$$

$$w_0 = w_1 = \frac{b-a}{2} w = \frac{1}{2}$$

$$t = \frac{b-a}{2} z + \frac{a+b}{2}$$

$$t_0 = \frac{b-a}{2} z_0 + \frac{a+b}{2} = \frac{1}{2}\left(\frac{-\sqrt{3}}{3}\right) + \frac{1}{2} = \frac{3-\sqrt{3}}{6} \qquad t_0 = \frac{3-\sqrt{3}}{6}$$

$$t_1 = \frac{b-a}{2} z_1 + \frac{a+b}{2} = \frac{1}{2}\left(\frac{\sqrt{3}}{3}\right) + \frac{1}{2} = \frac{3+\sqrt{3}}{6} \qquad t_1 = \frac{3+\sqrt{3}}{6}$$

⑥ a) If $n+1$ points Gaussian quadrature is used for numerical integration state the order of the polynomial that is integrated exactly.

If we take n as the number of points used; so that $\{z_i\}_{i=1}^{n}$

$$\text{Error} \propto f^{(2n)}(\mu)$$

So taking $n+1$ points:

$$\text{Error} \propto f^{(2(n+1))}(\mu)$$

$$\text{Error} \propto f^{(2n+2)}(\mu)$$

So polynomials up to $2n+1$ order are integrated exactly.

b) i) $\int_0^1 \sin x\,dx$ ii) $\int_0^1 x^3 dx$ iii) $\int_0^1 x^4 dx$ iv) $\int_0^1 x^{5.5} dx$.

$n=2 \rightarrow \text{Error} \propto f^{(6)}(\mu)$

i) $f^{(6)} \propto \sin x \neq 0$ ii) $f^{(6)} = 0$ iii) $f^{(6)} = 0$ iv) $f^{(6)} \neq 0$

No exact Exact Exact No exact

⑦ Compute $\int_0^1 12x\,dx$, $\int_0^1 (5x^3+2x)\,dx$

i) Trapezoidal (2 intervals) $x_0=0$ $x_1=0{,}5$ $x_2=1$

$\int_0^1 12x\,dx$

$$I = \frac{h}{2}\Big(f(x_0)+2f(x_1)+f(x_2)\Big) = \frac{0'5}{2}\Big(0+2\cdot 6+12\Big) = \frac{24}{4} = 6$$

$$\int_0^1 (5x^3+2x)\,dx \simeq \frac{h}{2}\Big(f(x_0)+2f(x_1)+f(x_2)\Big) = \frac{0'5}{2}\Big(0+2(1'625)+7\Big) = 2{,}5625$$

Trapezoidal Error:

$$E = -m\cdot\frac{h^3}{12} f^{2)}(\mu)$$

So the result for $\int_0^1 12x$ is exact. However, for $\int_0^1 (5x^3+2x)\,dx$ the result is not exact, with an error of $E = \left|2{,}5625 - \frac{9}{4}\right| = 0'3125$

$E_r = 13\%$

.) Simpson's rule (2 intervals) $x_0=0$ $x_1=0'25$ $x_2=0'5$ $x_3=0'75$ $x_4=1$

$$\int_0^1 12x\,dx \simeq \frac{h}{3}\Big[\Big(f(x_0)+4f(x_1)+f(x_2)\Big)+\Big(f(x_2)+4f(x_3)+f(x_4)\Big)\Big]$$

$$\int_0^1 12x\,dx \simeq \frac{0'25}{3}\Big[(0+4\cdot 3+6)+(6+4\cdot 9+12)\Big] = 6$$

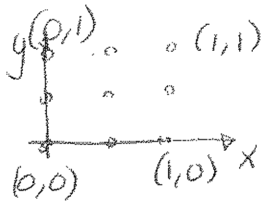
$$\int_0^1 (5x^3+2x)\,dx \simeq \frac{0'25}{3}\Big[\Big(0+4\cdot 0'5781+1'6250\Big)+\Big(1'6250+4\cdot 3'6094+7\Big)\Big] =$$

$$= 2{,}25$$

In this case, both integrals are exact due to the error that takes the following form:

$E = -\frac{m h^5}{\cdots} f^{(4)}(\mu)$

10 $\int_0^1\int_0^1 (9x^3+8x^2)(y^3+y)\,dx\,dy$. Simpson's rule.



Analytical

$$\int_0^1 (y^3+y)\left(\frac{9}{4}x^4+\frac{8}{3}x^3\right)\Big|_0^1 dy = \int_0^1 (y^3+y)\,\frac{59}{12}\,dy = \frac{59}{12}\left(\frac{y^4}{4}+\frac{y^2}{2}\right)\Big|_0^1 = \frac{177}{48} = 3{,}6875$$

Numerical

$$\int_0^1\int_0^1 (y^3+y)(9x^3+8x^2)\,dx\,dy \simeq \int_0^1 \frac{h}{3}\left(f(x_0,y)+4f(x_1,y)+f(x_2,y)\right)dy =$$

$$\boxed{\begin{aligned} f(x_0,y)&=0\\ f(x_1,y)&=\frac{25}{8}(y^3+y)\\ f(x_2,y)&=17(y^3+y)\end{aligned}}$$

$$= \int_0^1 \frac{1}{6}\left(0+4\cdot\frac{25}{8}+17\right)(y^3+y)\,dy =$$

$$= \int_0^1 \frac{1}{6}\left(\frac{59}{2}\right)\underbrace{(y^3+y)}_{f(y)}dy \simeq \frac{59}{12}\frac{1}{6}\left(f(y_0)+4f(y_1)+f(y_2)\right) = \frac{59}{72}\left(0+4\cdot\frac{5}{8}+2\right) = 3{,}6875$$

As simpson's rule integrates exactly third order polynomials, the obtained result is exact.