ORDINARY DIFFERENCIAL EQUATIONS EXERCISES. EMILIO SÁNCHEZ

PROBLEMA 1.

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \longrightarrow \frac{d^2\theta}{dt^2} = -g\theta$$

$$\Theta = [\Theta_{(1)} \quad \Theta_{(2)}]^T = [\theta \quad d\theta/dt]^T \rightarrow f(t,\Theta) = \frac{d\Theta}{dt} = [d\theta/dt \quad d^2\theta/dt^2]^T$$

$$f(t,\Theta) = [\Theta_{(2)} \quad -g\Theta_{(1)}]^T \quad ; \quad \text{I.C.} \rightarrow \Theta(1) = [0'4 \quad 0]^T$$

2 steps. $\Longrightarrow h = \Delta t = -0'5$

1./ $K_1 = f(t_0, \Theta_0) = [0 \quad -0'4g]^T$

$K_2 = f(t_0 + h, \Theta_0 + hK_1) = f([0'4 \quad g\,0'2]^T) = [0'2g \quad -0'4g]^T$

$$\boxed{\Theta_1 = \Theta_0} + \frac{h}{2}[K_1 + K_2] = [0'4 \quad 0]^T - 0'25[0'2g \quad -0'8g]^T =$$

$$= [0'4 - 0'05g \quad 0'2g]^T \boxed{\simeq [-0'09 \quad 1'96]^T}$$

2./ $K_1 = [0'2g \quad -0'4g + 0'05g^2]^T$

$K_2 = f([0'4 - 0'05g \quad 0'2g]^T - [0'1g \quad -0'2g + 0'025g^2]^T) = \cancel{[0'4g - 0'025g^2}$

$= [0'4g - 0'025g^2 \quad -0'4g + 0'15g^2]^T$

$$\boxed{\Theta_2} = \Theta_1 - 0'25[K_1 + K_2] = [0'4 - 0'05g \quad 0'2g]^T - 0'25[0'6g - 0'025g^2 \quad -0'8g + 0'2g^2]^T =$$

$$= [0'4 - 0'2g + 0'00625g^2 \quad 0'4g - 0'05g^2]^T \boxed{\simeq [-0'95975 \quad -0'882]^T}$$

4 STEPS $\Longrightarrow h = -0'25$

$$\Theta_1 = [0'4 - \frac{0'1}{8}g \quad 0'1g]^T \simeq [0'2775 \quad 0'98]^T$$

$$\Theta_2 = [0'4 - 0'05g + \frac{0'1}{256}g^2 \quad 0'2g - \frac{0'05}{8}g^2]^T \simeq [-0'0525 \quad 1'35975]^T$$

$$\Theta_3 = [0'4 - \frac{0'9}{8}g + \frac{0'9}{256}g^2 - \frac{0'1}{8192}g^3 \quad 0'3g - \frac{0'2}{8}g^2 + \frac{0'6}{2048}g^3]^T \simeq [-0'376 \quad 0'8147]^T$$

$$\Theta_4 = [0'4 - 0'2g + \frac{3'4}{256}g^2 - \frac{0'1}{512}g^3 + \frac{0'1}{8.32768}g^4 \quad 0'4g - \frac{0'5}{8}g^2 + \frac{1}{512}g^3 - \frac{0'1}{8192}g^4]^T \simeq [-0'4648 \quad -0'3568]^T$$

b) $$E_r = \frac{|-0'95975 + 0'4648|}{|-0'4648|} = \underline{\underline{1'065}}$$

c) 2^{nd} Order:

$$h^* = \left(\frac{10^{-3}}{1'065}\right)^{1/2+1} \cdot (-0'5) = \boxed{-0'049\ s}$$

PROBLEMA 2.

$$\frac{dy}{dx} = y - x^2 + 1 \;;\quad x \in (0,1) \;;\quad y(0) = y_0 = 1.$$

a) Euler with $h = 0'25$; $x_0 = 0$, $x_1 = 0'25$, $x_2 = 0'5$, $x_3 = 0'75$, $x_4 = 1$

$$y_{i+1} = y_i + hf(x_i; y_i) = y_i + 0'25\, f(x_i, y_i)$$

$y_1 = y_0 + 0'25 f(x_0, y_0) = 1 + 0'25(1+1) = 1'5$

$y_2 = y_1 + 0'25 f(x_1, y_1) = \dots = 2'109375$

$y_3 = y_2 + 0'25 f(x_2, y_2) = \dots = 2'8242$

$y_4 = y_3 + 0'25 f(x_3, y_3) = \dots = \boxed{3'6396}$

b) $C(h^{q+1}) = t$; Euler $q=1$; Heun $q=2$

$\not C(h^2) = \not C(h^{*3}) \Longrightarrow h^* = h^{2/3} \simeq 0'4$ (¿?)

Heun with $h = 0'5$.

1/ $K_1 = f(x_0, y_0) = 2$; $K_2 = f(0'5, y_0 + 0'5K_1) = f(0'5, 2) = 2'75$

$y_1 = y_0 + 0'25(K_1 + K_2) = 1 + 0'25 \cdot 4'75 = \underline{2'1875}$

2/ $K_1 = f(x_1, y_1) = f(0'5, 2'1875) = 2'9375$

$K_2 = f(1, 2'1875 + 0'5 \cdot 2'9375) = 3'65625$

$y_2 = y_1 + 0'25(K_1 + K_2) = 2'1875 + 0'25 \cdot 6'59375 = \underline{3'8359375}$

c) Aproximation $\longrightarrow p(x) = \sum_{i=0}^{n=2} a_i x^i$;

$x_0 = 0$; $x_1 = 0'5$; $x_2 = 1$

$y_0 = 1$; $y_1 = 2'1875$; $y_2 = 3'8359375$

$x_0 = 0$	1		
		$2'375$	
$x_1 = 0'5$	$2'1875$		$\frac{3'296875 - 2'375}{1} = 0'921875$
		$3'296875$	
$x_2 = 1$	$3'8359375$		

$p(x) = 1 + 2'375(x-0) + 0'921875(x-0'5)(x-0) = 1 + 2'375x - 0'4609375x + 0'921875x^2$;

$$\boxed{p(x) = 0'921875x^2 + 1'9140625x + 1}$$

d) Least squares fitting method.

x_i	0	$0'25$	$0'5$	$0'75$	1
$f(x_i)$	1	$1'5$	$2'109375$	$2'8242$	$3'6396$

$p(x) = Ax^2 + Bx + C = \sum_{i=0}^{2} a_i \Psi(x_i) = a_0\Psi_0(x) + a_1\Psi_1(x) + a_2\Psi_2(x)$

$\Psi_0(x) = 1$; $\Psi_1(x) = x$; $\Psi_2(x) = x^2$

Matricialmente:

$$\begin{bmatrix} \langle \varphi_0, \varphi_0 \rangle & \langle \varphi_0, \varphi_1 \rangle & \langle \varphi_0, \varphi_2 \rangle \\ \langle \varphi_1, \varphi_0 \rangle & \langle \varphi_1, \varphi_1 \rangle & \langle \varphi_1, \varphi_2 \rangle \\ \langle \varphi_2, \varphi_0 \rangle & \langle \varphi_2, \varphi_1 \rangle & \langle \varphi_2, \varphi_2 \rangle \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \langle \varphi_0, f \rangle \\ \langle \varphi_1, f \rangle \\ \langle \varphi_2, f \rangle \end{bmatrix}$$

Sabiendo que: $\langle f, g \rangle = \sum_{i=0}^{4} f(x_i)g(x_i)$

$\langle \varphi_0, \varphi_0 \rangle = \sum_{i=0}^{4} 1 = 5$; $\langle \varphi_0, \varphi_1 \rangle = \langle \varphi_1, \varphi_0 \rangle = \sum_{i=0}^{4} x_i = 2'5$

$\langle \varphi_0, \varphi_2 \rangle = \langle \varphi_2, \varphi_0 \rangle = \sum_{i=0}^{4} x_i^2 = 1'875 = \langle \varphi_1, \varphi_1 \rangle$

$\langle \varphi_1, \varphi_2 \rangle = \langle \varphi_2, \varphi_1 \rangle = \sum_{i=0}^{4} x_i \cdot x_i^2 = \sum_{i=0}^{4} x_i^3 = 1'5625$

$\langle \varphi_2, \varphi_2 \rangle = \sum_{i=0}^{4} (x_i^2)^2 = 1'3828125$; $\langle \varphi_0, f \rangle = \sum_{i=0}^{4} f_i = 11'073175$

$\langle \varphi_1, f \rangle = \sum_{i=0}^{4} x_i f_i = 7'1874375$; $\langle \varphi_2, f \rangle = \sum_{i=0}^{4} x_i^2 f_i = 5'84930625$

$$\begin{bmatrix} 5 & 2'5 & 1'875 \\ 2'5 & 1'875 & 1'5625 \\ 1'875 & 1'5625 & 1'3828125 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 11'073175 \\ 7'1874375 \\ 5'84930625 \end{bmatrix} \Longrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0'9991 \\ 1'7999 \\ 0'8414 \end{bmatrix}$$

Resultando que: $\boxed{p(x) = 0'8414x^2 + 1'7999x + 0'9991}$

Esta última aproximación es peor que la obtenida en el apartado c, a pesar de estar realizada con más datos y con un método más preciso. ~~E~~ No obstante, dichos datos tienen mayor error respecto a la solución analítica, de hecho, el polinomio obtenido es bastante preciso en cuanto a los datos introducidos para obtenerlo. Este es el motivo por el que es peor solución que el obtenido en c).

PROBLEMA 3:

a) $$\frac{dy}{dx} = f(x,y) \quad ; \quad x \in (0,1) \; ; \; y(0) = 1$$

Taylor: $y_{i+1} = y_i + h \frac{dy}{dx}(x_i) + O(h^2)$

$$\frac{dy}{dx} = f(x,y) \Longrightarrow y_{i+1} = y_i + h f(x_i, y_i) + O(h^2)$$

$$O(h^2) = y_{i+1} - y_i - h f(x_i, y_i) \longrightarrow \frac{O(h^2)}{h} = \boxed{\tau_i(h) = \frac{y_{i+1} - y_i}{h} - f(x_i, y_i)}$$

Condición de consistencia:

$$\text{máx} \left\{ \lim_{h \to 0} \tau_i(h) = 0 \right\} \quad ; \; i = 0, \dots, m$$

$$\lim_{h \to 0} \tau_i(h) = \lim_{h \to 0} \left(\frac{y_{i+1} - y_i}{h} - f(x_i, y_i) \right) = \frac{0}{0} \text{ Indeterminación.}$$

Aplicando L'HÔPITAL:

$$\lim_{h \to 0} \tau_i(h) = \frac{0}{0} = \lim_{h \to 0} \frac{0}{1} - 0 = 0 \quad \#$$

El método es consistente.

b) Backward Euler:

Taylor: $y_i = y_{i+1} - h \frac{dy}{dx}(x_{i+1}) + O(h^2)$

$$\frac{dy}{dx}(x_{i+1}) = \frac{y_{i+1} - y_i}{h} + \tau_i(h) \quad ; \text{ Siendo } \tau_i(h) = \frac{O(h^2)}{h}$$

Reemplazando en la ecuación diferencial:

$$y_{i+1} = y_i + h f(x_{i+1}, y_{i+1}) + h \tau_i(h)$$

Para el esquema numérico se desprecia el error de truncamiento ($\tau_i(h)$), resultando:

$$\boxed{Y_{i+1} = Y_i + h f(x_{i+1}, Y_{i+1}) \quad ; \; i = 0, \dots m-1}$$

c) Stability for Forward Euler: $f(x,y) = -\lambda y$

$$Y_{i+1} = Y_i - h\lambda Y_i = (1 - h\lambda) Y_i$$

Condición de estabilidad:

$$|1 - h\lambda| < 1 \longrightarrow \lambda \in \mathbb{R}^+ \Longrightarrow 1 - \lambda h < 1 \longrightarrow \boxed{\lambda h > 0}$$

Backward Euler: $Y_{i+1} = Y_i - \lambda h Y_{i+1} \longrightarrow (1 + \lambda h) Y_{i+1} = Y_i$

$$|1 + \lambda h| > 1 \longrightarrow \lambda \in \mathbb{R}^+ \Longrightarrow 1 + \lambda h > 1 \Longrightarrow \boxed{\lambda h > 0}$$

En ambos casos los métodos son incondicionalmente estables para $\lambda \in \mathbb{R}^+$

d) $\frac{dy}{dx} = -25y^{3'5}$; $y(0) = 1$; $h = \frac{1}{10}$

$$Y_1 = Y_0 + \frac{1}{10} f\left(\frac{1}{10}, Y_1\right) = 1 + \frac{1}{10}\left(-25\, Y_1^{3'5}\right)$$

$g(Y_1) = Y_1 + 2'5\, Y_1^{3'5} - 1 = 0$; Aproximación inicial despreciando el término Y_1 :

$$Y_1^k = 0'769667 \quad ; \quad \frac{dg}{dY_1} = 8'75\, Y_1^{2'5} + 1$$

~~Iteración 1:~~ $Y_1^{k+1} = Y_1^k - \frac{g(Y_1^k)}{g'(Y_1^k)} = 0'769667 - \frac{0'769667}{5'5474} = \underline{0'63092}$

$$Y_1^{k+2} = Y_1^{k+1} - \frac{g(Y_1^{k+1})}{g'(Y_1^{k+1})} = \underline{0'596504}$$

$Y_2 = Y_1 + \frac{1}{10} f\left(\frac{2}{10}, Y_2\right) = 0'596504 + \frac{1}{10}\left(-25\, Y_2^{3'5}\right)$;

$g_2(Y_2) = Y_2 + 2'5\, Y_2^{3'5} - Y_1^{k+2} = 0 \Longrightarrow \underline{Y_2^k \approx 0'66404}$

$$Y_2^{k+1} = Y_2^k - \frac{g_2(Y_2^k)}{g_2'(Y_2^k)} = \underline{0'503798}$$

$$Y_2^{k+2} = Y_2^{k+1} - \frac{g_2(Y_2^{k+1})}{g_2'(Y_2^{k+1})} = \underline{0'451710}$$

e) Forward Euler : $Y_1 = Y_0 + \frac{1}{10} f(x_0, y_0) = 1 + \frac{1}{10}(-25) = \underline{-1'5}$

$Y_2 = Y_1 + \frac{1}{10} f(h, Y_1) = -1'5 + \frac{1}{10}\left(-25\left(-1'5^{3'5}\right)\right) = \cancel{8'833785}$ ¿?

$(-1'5)^{3'5}$ No tiene solución para el conjunto de los reales.
$\Rightarrow$ Stability problems.

f) Mediante la solución analítica: Condición estabilidad: $\lambda h > 0$
siendo $\lambda = 25y^{2'5}$

$x_1 = \frac{1}{10} \longrightarrow y_1 = 0'45276$; $x_2 = \frac{2}{10} \longrightarrow y_2 = 0'3530746$

En $x = 1 \longrightarrow \boxed{y(1) = 0'1900599}$ (valor de comparación).

$h = 1/15 \longrightarrow y(1) = -0'063328$ X

$h = 1/30 \longrightarrow y(1) = 0'134987 \longrightarrow y\left(\frac{1}{10}\right) = 0'163568$ X

$h = 1/45 \longrightarrow y(1) = 0'181812 \longrightarrow y\left(\frac{1}{10}\right) = 0'366973$ ~

$h = 1/90 \longrightarrow y(1) = 0'187407 \longrightarrow y(1/10) = 0'424597$ ✓.

Para $\underline{h = 1/45}$ el valor obtenido tiene cierta estabilidad, no obstante, presenta una ligera discontinuidad para el valor de x próximo a 0. Todos los intervalos así lo presentan, no obstante, cuanto menor es h, menor es el efecto o error de esta discontinuidad.

PROBLEMA 4.

$$\frac{d^2y}{dx^2} + \omega^2 y = 0 \ ; \quad x \in (0,1) \ ;$$

$$B.C.: y(0) = 0 \ ; \quad dy/dx(0) = \omega .$$

a) $y = [y(1) \quad y(2)]^T = [y \quad dy/dx]^T$

$f(x,y) = dy/dx = [dy/dx \quad d^2y/dx^2]^T = [y(2) \quad -\omega^2 y(1)]^T$

B.C.: $y_0 = [0 \quad \omega]^T$

b) Solution at $x=1$ with 4 steps $\Rightarrow h = \Delta x = 0'25$

1: $y_1 = y_0 + h f(x_0, y_0) = [0 \quad \omega]^T + 0'25[\omega \quad 0]^T = [0'25\omega \quad \omega]^T$

2: $y_2 = y_1 + hf(x_1, y_1) = [0'25\omega \quad \omega]^T + [0'25\omega \quad -0'25^2\omega^3]^T = [0'5\omega \quad \omega - 0'25^2\omega^3]^T$

3: $y_3 = y_2 + hf(x_2, y_2) = [0'5\omega \quad \omega - 0'25^2\omega^3]^T + [0'25\omega - 0'25^3\omega^3 \quad -2\cdot 0'25^2\omega^3]^T =$
$= [3\cdot 0'25\omega - 0'25^3\omega^3 \quad \omega - 3\cdot 0'25^2\omega^3]^T$

4: $y_4 = y_3 + hf(x_3, y_3) = [3\cdot 0'25\omega - 0'25^3\omega^3 \quad \omega - 3\cdot 0'25^2\omega^3]^T +$
$+ [\cancel{\omega}\,0'25\omega - 3\cdot 0'25^3\omega^3 \quad -3\cdot 0'25^2\omega^3 + 0'25^4\omega^5]^T =$
$= [\omega - 0'25^2\omega^3 \quad \omega - 6\cdot 0'25^2\omega^3 + 0'25^4\omega^5]^T$

Para $\omega^2 = 3 \longrightarrow y_4 = [1'40729 \quad -0'155614]^T$

Mediante Matlab: $y(1) = [\underline{\underline{1'40729128}} \quad -0'1556139397 4]^T$ #

c) with 8 steps: $y(1) = [\underline{\underline{1'1901836705}} \quad -0'27984829 66]^T$

$$E_r(h=0'25) = \frac{|1'40729 - 1'190181|}{1'19018} = \underline{0'182415}$$

$$h^* = \left(\frac{10^{-2*}}{0'182415}\right)^{1/2} \cdot 0'25 = 0'0585 \Longrightarrow 17'08 \rightarrow \underline{18 \text{ steps}}$$

* 10^{-2}: 3 cifras significativas $\Rightarrow$ la unidad y 2 decimales.

El valor obtenido es: $y(1) = 1'073304\ldots = \underline{\underline{1'07}}$

No obstante, el método tiene una mala convergencia, realizando algunas pruebas sobre el código, parece que el valor final converge ~~en~~, con 3 dígitos significativos, en:

$$\underline{\underline{y(1) = 0'987}}$$