FINITE Element Method

HOME WORK - 2

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Solution:-

Plane Elasticity:

Here we are analysing plane elasticity problem of prismatic bodies, assuming plane stress.

Given data:-

thickness, $t = 1\,m$

Young's Modulous, $E = 10\,GPa$

Poisson Ratio, $\nu = 0.2$

Vertical Displacement, $\delta = 10^{-2} = 0.01\,m$

Body force $= \rho g = 10^{3}\,N/m^2$

> Strong form

Strong form is written as,

$$b_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \qquad \text{---} \qquad (1)$$

$$b_y + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \qquad \text{---} \qquad (2)$$

> Boundary Conditions

Boundary conditions are through displacement of given nodes in x & y direction.

From the fig. given, it is clear that,

$$u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0 \quad (\text{Fixed nodes } 1, 2, 3)$$

$$u_5 = 0 \quad (\text{Symmetry Condition})$$

$$\delta = 10^{-2} = 0.01\,m = v_6 \quad (\text{Given data})$$

> Nodal Co-ordinates (X) & Connectivity Matrix (T):

NODES	X	Y
1	-3	0
2	-1.5	0
3	0	0
4	-1.5	1.5
5	0	1.5
6	0	3

Table ①: Nodal Co-ordinates (X)

	Nodes		
Element	1	2	3
1	2	4	1
2	4	2	5
3	3	5	2
4	5	6	4

Table ②: T-Matrix [Connectivity Matrix]

> Description of Mesh:

From fig, we have four elements in order to make the descritization easier local numbering is made, such that in every element, the node ~~eg~~ in the right angle vertex has a local number equal to 1, which is shown in above figure.

Now, to find the descritization of displacement field, we have.

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

The three nodes of triangular mesh defines linear displacement field which can be written as.

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

After deriving the shape functions for 'u' alone, we get

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$
$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$
$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$N_i = \frac{1}{2A^{(e)}} (a_i + b_i x + c_i y) \longrightarrow (3)$$

where,

$$a_i = x_j x_k - x_k x_j \; ; \quad b_i = y_j - y_k \; ;$$
$$c_i = x_k - x_j$$

We know that the stiffness matrix is given by,

$$K^e = \iint_{A^e} B^T D B t \, dA \longrightarrow (4)$$

Equivalent Nodal force Vector,

$$f^e = f^e_\epsilon + f^e_\sigma + f^e_b + f^e_t \longrightarrow (5)$$

$$f^e_\epsilon = \iint_{A^e} B^T D \epsilon^0 t \, dA$$

$$f^e_\sigma = \iint_{A^e} B^T D \sigma^0 t \, dA$$

$$f^e_b = \iint_{A^e} N^T \mathbf{b} \, t \, dA$$

$$f^e_t = \iint_{A^e} N^T \mathbf{t} \, t \, dA$$

Eq^n (4) Can also be written as,

$$K_{ij} = \left(\frac{t}{4A}\right)^e \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix} \rightarrow (6)$$

As we are dealing with plane stress problem, we have

$$\sigma = D \epsilon$$

D is the constitutive matrix defined for given data as,

$$D = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

$$d_{11} = d_{22} = \frac{E}{(1-\nu^2)} = \frac{10 \, GPa}{(1-0.2^2)} = 10.417 \, GPa$$

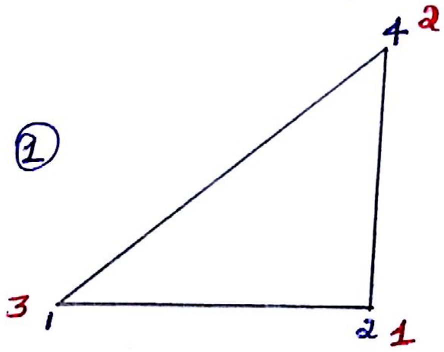
$$d_{12} = d_{21} = \nu d_{11} = 0.2 \times (10.417) = 2.083 \, GPa$$

$$d_{33} = \frac{E}{2(1+\nu)} = \frac{10}{2(1+0.2)} = 4.167 \text{ GPa}$$

> To Compute Stiffness Matrix for Elements 1, 3 & 4

From Nodal Co-ordinates & Considering local numbering, we have

Element ②



1, 2, 4 → Global Numbering
1, 2, 3 → Local Numbering

· From fig ①: Element ①

$$(x_1, y_1)^1 = (-1.5, 0)$$

$$(x_2, y_2)^1 = (-1.5, 1.5)$$

$$(x_3, y_3)^1 = (-3, 0)$$

But,

$$b_i = y_j - y_k \quad \& \quad c_i = x_k - x_j$$

$$b_1 = y_2 - y_3 = 1.5 \quad ; \quad c_1 = x_3 - x_2 = -1.5$$

$$b_2 = y_3 - y_1 = 0 \quad ; \quad c_2 = x_1 - x_3 = 1.5$$

$$b_3 = y_1 - y_2 = -1.5 \quad ; \quad c_3 = x_2 - x_1 = 0$$

Now using Eqn ⑥, we have

$$K_{ij} = \left(\frac{t}{4A}\right)^e \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$$

$$K_{11}^{①} = \frac{2}{9}\begin{bmatrix} 2.25\times10.417 + 2.25\times4.167 & -2.25\times2.083 - 2.25\times4.167 \\ -2.25\times2.083 + (-2.25)\times4.167 & 2.25\times10.417 + 2.25\times4.167 \end{bmatrix}$$

$$K_{11}^{①} = \frac{2\times2.25}{9}\begin{bmatrix} 10.417+4.167 & -2.083-4.167 \\ -2.083-4.167 & 10.417+4.167 \end{bmatrix}$$

$$K_{11}^{①} = \begin{bmatrix} 7.292 & -3.125 \\ -3.125 & 7.292 \end{bmatrix}$$

Similarly

$$K_{12}^{①} = \frac{2\times2.25}{9}\begin{bmatrix} -4.167 & 2.083 \\ 4.167 & -10.417 \end{bmatrix} = \begin{bmatrix} -2.083 & 1.042 \\ 2.083 & -5.2085 \end{bmatrix} = K_{21}^{e}$$

$$K_{13}^{①} = \frac{2\times2.25}{9}\begin{bmatrix} -10.417 & 4.167 \\ 2.083 & -4.167 \end{bmatrix} = \begin{bmatrix} -5.2085 & 2.083 \\ 1.042 & -2.083 \end{bmatrix} = K_{31}^{e}$$

$$K_{22}^{①} = \frac{2\times2.25}{9}\begin{bmatrix} 4.167 & 0 \\ 0 & 10.417 \end{bmatrix} = \begin{bmatrix} 2.083 & 0 \\ 0 & 5.2085 \end{bmatrix}$$

$$K_{23}^{①} = \frac{2\times2.25}{9}\begin{bmatrix} 0 & -4.167 \\ -2.083 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2.083 \\ -1.042 & 0 \end{bmatrix} = K_{32}^{e}$$

$$K_{33}^{①} = \frac{2\times2.25}{9}\begin{bmatrix} 10.417 & 0 \\ -2.083 & 4.167 \end{bmatrix} = \begin{bmatrix} 5.2085 & 0 \\ 0 & 2.083 \end{bmatrix}.$$

Since Nodal Co-ordinates & local co-ordinates are same for elements ①, ③ & ④, we have

$$K_{11}^{③} = K_{11}^{④} = K_{11}^{①} \; ; \; K_{12}^{③} = K_{12}^{④} = K_{12}^{①}$$
$$K_{13}^{③} = K_{13}^{④} = K_{13}^{①}$$

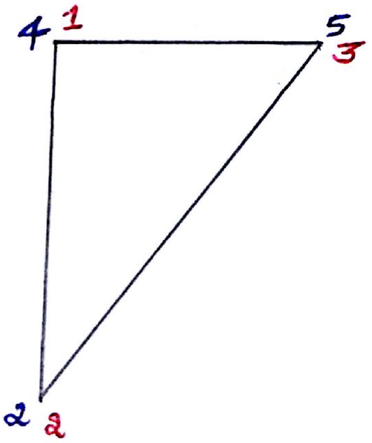
$$K_{21}^{③} = K_{21}^{④} = K_{21}^{①} \; ; \; K_{22}^{③} = K_{22}^{④} = K_{22}^{①}$$
$$K_{23}^{③} = K_{23}^{④} = K_{23}^{①}$$

$$K_{31}^{③} = K_{31}^{④} = K_{31}^{①} \; ; \; K_{32}^{③} = K_{32}^{④} = K_{32}^{①}$$
$$K_{33}^{③} = K_{33}^{④} = K_{33}^{①}$$

Therefore element 1, 3 & 4 have same stiffness matrix

> To Compute Stiffness Matrix for element ②

From Nodal Coordinates and considering local numbering, we have



2, 4, 5 → Global Numbering
1, 2, 3 → Local Numbering

Figure ②: Element 2

$$(x_1, y_1)^2 = (-1.5, +1.5)$$

$$(x_2, y_2)^2 = (-1.5, 0)$$

$$(x_3, y_3)^2 = (0, +1.5)$$

W.K.T

$$b_i = y_j - y_k \quad \& \quad c_i = x_k - x_j$$

$$b_1 = y_2 - y_3 = -1.5 \quad ; \quad c_1 = x_3 - x_2 = 1.5$$

$$b_2 = y_3 - y_1 = 0 \quad ; \quad c_2 = x_1 - x_3 = -1.5$$

$$b_3 = y_1 - y_2 = 1.5 \quad ; \quad c_3 = x_2 - x_1 = 0$$

Since the values are same as that of element 1, but opposite in sign. Hence we have the same stiffness matrix for the element 2 as well,

Therefore, $K^1 = K^2 = K^3 = K^4$

> Assembling of Stiffness Matrix

$$K^1 = \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 \end{bmatrix} = K^2 = \begin{bmatrix} K_{11}^2 & K_{12}^2 & K_{13}^2 \\ K_{21}^2 & K_{22}^2 & K_{23}^2 \\ K_{31}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix} =$$

$$K^3 = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 \end{bmatrix} = K^4 = \begin{bmatrix} K_{11}^4 & K_{12}^4 & K_{13}^4 \\ K_{21}^4 & K_{22}^4 & K_{23}^4 \\ K_{31}^4 & K_{32}^4 & K_{33}^4 \end{bmatrix}$$

The assembly of matrix is shown below :-

$$\mathbf{K} = \begin{bmatrix} K_{33}^{(1)} & K_{13}^{(1)T} & 0 & K_{23}^{(1)} & 0 & 0 \\ & K_{11}^{(1)} + K_{22}^{(2)} + K_{33}^{(3)} & K_{13}^{(3)T} & K_{12}^{(1)} + K_{12}^{(2)T} & K_{23}^{(2)} + K_{23}^{(3)T} & 0 \\ & & K_{11}^{(3)} & 0 & K_{12}^{(3)} & 0 \\ & & & K_{22}^{(1)} + K_{11}^{(2)} + K_{33}^{(4)} & K_{13}^{(2)} + K_{13}^{(4)T} & K_{23}^{(4)T} \\ & & & & K_{11}^{(4)} + K_{22}^{(3)} + K_{33}^{(2)} & K_{12}^{(4)} \\ & & & & & K_{22}^{(4)} \end{bmatrix}$$

We know that,

$$a_i^e = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \Rightarrow \quad a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

for the given problem, the displacement matrix '**a**' is given by

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

And the Nodal force Vector is given by

$$f = \begin{bmatrix} f_3^{(1)} + r_1 \\ f_1^{(1)} + f_2^{(2)} + f_3^{(3)} + r_2 \\ f_1^{(3)} + r_3 \\ f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ f_3^{(2)} + f_2^{(3)} + f_1^{(4)} \\ f_2^{4} \end{bmatrix}$$

therefore,

$$Ka = f$$

* The given problem has 12 degrees of freedom, where in 9 degrees are constrained.

> To Compute Nodal displacements :-

From boundary conditions, we have,

$u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0$;

hence $a_1 = a_2 = a_3 = 0$

Hence we need to consider only 4, 5 & 6 rows only from global stiffness matrix.

and also $u_5 = u_6 = 0$

Therefore,

$$\begin{bmatrix} K_{22}^{(1)} + K_{11}^{(2)} + K_{33}^{(4)} & K_{13}^{(2)} + K_{13}^{(4)T} & K_{23}^{(4)T} \\ & K_{11}^{(4)} + K_{22}^{(3)} + K_{33}^{(2)} & K_{12}^{(4)} \\ & & K_{22}^{(4)} \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \\ u_5 = 0 \\ v_5 \\ u_6 = 0 \\ v_6 = \delta = 10^{-2} \end{bmatrix} = \begin{bmatrix} f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ f_3^{(2)} + f_2^{(3)} + f_1^{(4)} \\ f_2^{(4)} \end{bmatrix} \longrightarrow (7)$$

Now substituting value of K_{ij}^{e} for the above matrix, we get,

$$K = \begin{bmatrix} -14.5835 & -3.125 & -10.417 & 3.125 & 0 & 1.042 \\ -3.125 & 14.5835 & 4.166 & -4.166 & -1.042 & 0 \\ -10.417 & 4.166 & 14.5835 & -3.125 & -2.083 & 1.042 \\ 3.125 & -4.166 & -3.125 & 14.5835 & 2.083 & -5.2085 \\ 0 & -1.042 & -2.083 & 2.083 & 2.083 & 0 \\ 1.042 & 0 & 1.042 & -5.2085 & 0 & 5.2085 \end{bmatrix} \text{ GN/m}$$

We have from data that the whole domain deforms because of self weight with gravity acting in the direction y-axis. Therefore only body forces are significant and no surface loads.

Body forces for the equivalent nodal force is given by,

$$f_{b_i} = \left(\frac{At}{3}\right)^{e} \begin{bmatrix} b_x \\ b_y \end{bmatrix} \longrightarrow (8)$$

but,

$$b_x = 0 \quad ; \quad b_y = -\rho g = -10^3$$

$$f_{b_i} = \left(\frac{2.25}{6}\right)\begin{bmatrix} 0 \\ -10^3 \end{bmatrix} = \begin{bmatrix} 0 \\ -375 \end{bmatrix} N$$

Substituting $f_{b_i}^e$ & K in eqn ⑦ we get by simplifying,

$$\begin{bmatrix} 14.5835 & -3.125 & 3.125 & 1.042 \\ -3.125 & 14.5835 & -4.166 & 0 \\ 3.125 & -4.166 & 14.5835 & -5.2085 \\ 1.042 & 0 & -5.2085 & 5.2085 \end{bmatrix} \times 10^9 \begin{bmatrix} u_4 \\ v_4 \\ v_5 \\ -10^{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1125 \\ -1125 \\ -375 \end{bmatrix}$$

When we solve the above system of linear equations,
we get the nodal displacement values as;

$$\boxed{\begin{aligned} u_4 &= -1.29 \times 10^{-4}\, m \\ v_4 &= -1.13 \times 10^{-3}\, m \\ v_5 &= -3.87 \times 10^{-3}\, m \end{aligned}}$$