FINITE ELEMENTS

Homework 1

Consider the following differential equation

$$-u'' = f \text{ in }]0,1[$$

with the boundary conditions $u(0) = 0$ and $u(1) = \alpha$.

The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0, 1, \ldots, n$ and $h = 1/n$.

1. Find the weak form of the problem. Describe the FE approximation u^h.
2. Describe the linear system of equations to be solved.
3. Compute the FE approximation u^h for $n = 3$, $f(x) = \sin x$ and $\alpha = 3$. Compare it with the exact solution, $u(x) = \sin x + (3 - \sin 1)x$.

1. Find the weak form of the problem. Describe the FE approximation u^h.

$$-\frac{d^2u}{dx^2} = f \rightarrow \frac{d^2u}{dx^2} + f = 0$$

Introducing a weight function w which satisfies: $w(0)= w(1)=0$ and integration in the domain]0,1[:

$$\int_\Omega w \frac{d^2u}{dx^2}\, dx + \int_\Omega w\, f\, dx = 0$$

Then integrating by parts the first term of the equation:

$$\left[w \frac{du}{dx}\right]_0^1 - \int_0^1 \frac{dw}{dx}\frac{du}{dx}\, dx + \int_0^1 w\, f\, dx = 0$$

As we have defined w to be $w(0)=w(1)=0$, the first term of the equation cancels out, yielding the **Weak Form** of the partial differential equation:

$$\int_0^1 \frac{dw}{dx}\frac{du}{dx}\, dx = \int_0^1 w\, f\, dx$$

2. Describe the linear system of equations to be solved.

To approximate the function u we will use the linear interpolation:

$$u \sim u^h = \sum_{j=1}^{n} N_j a_j$$

$$\frac{du^h}{dx} = \sum_{j=1}^{n} N_j' a_j$$

Where N_j is the j component of the vector field of shape functions, and a_j is the coefficient that multiplies the shape function of the element j. Then substituting:

$$\int_0^1 \frac{dw}{dx}\frac{du^h}{dx}\,dx = \sum_{j=1}^{n}\int_0^1 \frac{dw}{dx} N_j{}' a_j\,dx = \int_0^1 w\,f\,dx$$

Now we have to define the weight function w. Using Galerkin Method, we choose:

$$w = N_i\ ;\ \frac{dw}{dx} = N_i{}'$$

So the expression yields:

$$\sum_{j,i=1}^{n}\int_0^1 N_i{}' N_j{}' a_j\,dx = \sum_{i=1}^{n}\int_0^1 N_i\,f\,dx$$

What defines a linear system of equations where the unknowns are the coefficients $a_j$that satisfies the weak form of the PDE. In matrix form, for one single 2-noded element using local indexing, the system takes the form:

$$\begin{bmatrix} N_1{}'N_1{}' & N_1{}'N_2{}' \\ N_2{}'N_1{}' & N_2{}'N_2{}' \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} N_1\,f \\ N_2\,f \end{bmatrix}$$

$$K_{ij} = \int_0^{l^e} N_i{}'N_j{}'\,dx$$

$$f_i = \int_0^{l^e} N_i\,f\,dx$$

3. **Compute the FE approximation u^h for n=3, $f(x) = sin(x)$ and $\alpha = 3$. Compare it with the exact solution, $u(x) = \sin x + (3 - \sin 1)x$.**

We choose the shape and weight function defined in local indexing for each element:

$$N_1 = \frac{x_2{}^e - x}{l^e};\ N_2 = \frac{x - x_1{}^e}{l^e}$$

$$N_1' = -\frac{1}{l^e};\ N_2' = \frac{1}{l^e}$$

where l^e is the length of the element

Then we have to compute the contribution of each element to the FEM:

$$K_{11} = K_{22} = K_{33} = K_{44} = \int_0^{l^e} N_1{}'N_1{}'\,dx = \int_0^{l^e} \frac{1}{(l^e)^2}\,dx = \left(\frac{l^e}{(l^e)^2}\right) - \left(\frac{0}{(l^e)^2}\right) = \frac{1}{l^e} = 3$$

$$K_{12} = K_{21} = K_{32} = K_{23} = K_{43} = K_{34} = \int_0^{l^e} N_2{}'N_1{}'\,dx = \int_0^{l^e} \frac{-1}{(l^e)^2}\,dx = \left(\frac{-l^e}{(l^e)^2}\right) - \left(\frac{0}{(l^e)^2}\right) = -\frac{1}{l^e} = -3$$

$$f_1{}^1 = \int_0^{1/3} N_1\,f\,dx = \int_0^{1/3} \frac{x_2{}^e - x}{l^e}\,sin(x)\,dx = \left[\frac{(-cos(x)\,x_2^1 - sin(x) + x\,cos(x))}{l^e}\right]_0^{1/3} = 0.018415909611543$$

$$f_2{}^1 = \int_0^{1/3} N_2\, f\, dx = \int_0^{1/3} \frac{x - x_1{}^e}{l^e}\, sin(x)\, dx = \left[\frac{(cos(x)\, x_1^1 + sin(x) - x\, cos(x))}{l^e}\right]_0^{1/3} = $$
$$= -0.018415909611543$$

$$f_1{}^2 = \left[\frac{(-cos(x)\, x_2^2 - sin(x) + x\, cos(x))}{l^e}\right]_{1/3}^{2/3} = \ 0.071431627493983$$

$$f_2{}^2 = \left[\frac{(cos(x)\, x_1^2 + sin(x) - x\, cos(x))}{l^e}\right]_{1/3}^{2/3} = \ -0.071431627493983$$

$$f_1{}^3 = \left[\frac{(-cos(x)\, x_2^3 - sin(x) + x\, cos(x))}{l^e}\right]_{2/3}^{1} = 0.116583715562470$$

$$f_2{}^3 = \left[\frac{(cos(x)\, x_1^3 + sin(x) - x\, cos(x))}{l^e}\right]_{2/3}^{1} = \ -0.116583715562470$$

$$3\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & (1+1) & -1 & 0 \\ 0 & -1 & (1+1) & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0.0184 + 0 \\ 0.0714 - 0.0184 \\ -0.0714 + 0.1166 \\ -0.1166 + 3 \end{bmatrix} = \begin{bmatrix} 0.018415909611543 \\ 0.053015717882440 \\ 0.045152088068487 \\ 2.883416284437530 \end{bmatrix}$$

As we have Dirichlet boundary conditions in x=0 and x=1, the value of the functions at these points ($a_1 = u(0) = 0$), ($a_4 = u(1) = \alpha$) are not unknowns. Therefore we must simplify the linear system to solve in the following way:

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.0530 - (-3 \cdot 0) \\ 0.0451 - (-3\alpha) \end{bmatrix} = \begin{bmatrix} 0.053015717882440 \\ 9.045152088068488 \end{bmatrix}$$

$$a = \begin{bmatrix} 1.016798169314819 \\ 2.015924432668824 \end{bmatrix}$$

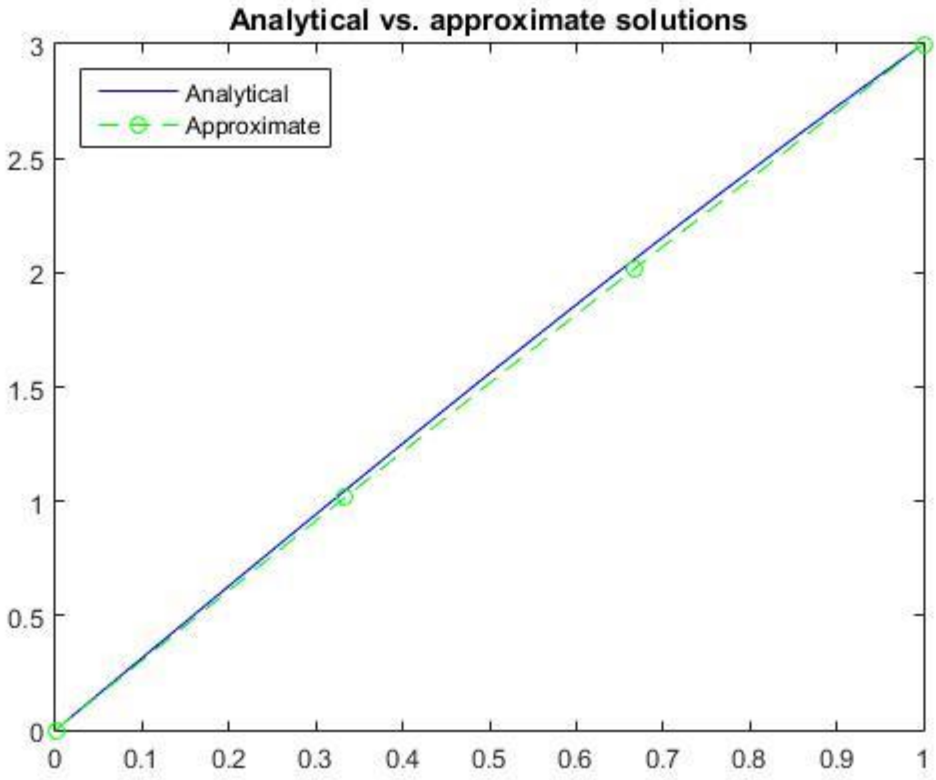
So, all the values of the function calculated with the FEM are:

$$YFEM = \begin{bmatrix} 0 \\ 1.016798169314819 \\ 2.015924432668824 \\ 3 \end{bmatrix}$$

And the analytical solution at the same points is:

$$YANA = \begin{bmatrix} 0 \\ 1.046704368526853 \\ 2.057389146531139 \\ 3 \end{bmatrix}$$

As it is seen, the approximate solution is very close to the analytical one with a discretization of only 4 nodes, and of course at the boundaries where the values are prescribed, the solution coincides. To show better how good is the approximate solution to the analytical one see **Graph 1**.



Graph 1: *Comparison between the approximate solution and the analytical one.*