Finite Elements - Homework 2

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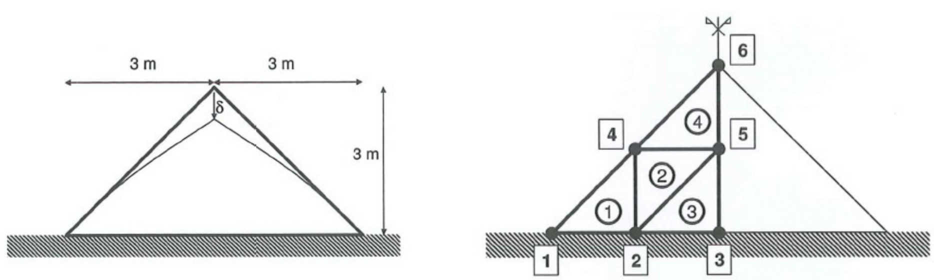
A triangular thin plate is deformed under its self weight and an imposed vertical displacement δ on the tip. A plane stress model is used to analyze the structural response of the plate. The thickness is assumed to be equal to 1. Using the geometry of the problem, only the left half of the domain is analyzed.

1. Describe the strong form of the problem in the reduced domain (left half). Indicate accurately the Boundary Conditions in every edge
2. Describe the mesh by giving the arrays of nodal coordinates X and the connectivity matrix T. In order to simplify the calculations select the local numbering of the nodes, such that, in every element, the node in the right angle vertex has local number equal to 1.
3. Set up the linear system of equations corresponding to the discretization shown in figure 2. How many degrees of freedom has the system to be solved?
4. Compute the FE approximation u^h. Use $G = 10GPa, \nu = 0{,}2, \delta = 10^{-2}m$ and $pg = 10^3 N/m^2$.



(a) Geometry of the domain

(b) Mesh description

1 Describe the strong form of the problem in the reduced domain (left half). Indicate accurately the Boundary Conditions in every edge

Equilibrium equation:

$$div \sigma + b = 0 \tag{1}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} + \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

Constitutive equation:

$$\sigma = D\epsilon \tag{3}$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = E \begin{pmatrix} \frac{1}{1-\nu^2} & \frac{\nu}{1-\nu^2} & 0 \\ \frac{\nu}{1-\nu^2} & \frac{1}{1-\nu^2} & 0 \\ 0 & 0 & \frac{1}{2(1+\nu)} \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{4}$$

Compatibility equation:

$$\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} \tag{5}$$

<u>Strong Form:</u>

$$\begin{pmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \underline{D} \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \underline{u} + \underline{b} = \underline{0} \tag{6}$$

<u>Boundary Conditions:</u>

Nodes 1 to 3 are fixed:

$$\begin{pmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7}$$

In node 6, a vertical displacement $\delta = -0{,}01$ is imposed.

$$u_{6y} = \delta = -0{,}01 \tag{8}$$

Finally, because of the simmetry of the problem:

$$\begin{pmatrix} u_{5y} \\ u_{6y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

2 Describe the mesh by giving the arrays of nodal coordinates X and the connectivity matrix T. In order to simplify the calculations select the local numbering of the nodes, such that, in every element, the node in the right angle vertex has local number equal to 1.

$$X = \begin{pmatrix} 1{,}5 & 1{,}5 & 0 \\ 1{,}5 & 1{,}5 & 3 \\ 3 & 3 & 1{,}5 \\ 3 & 3 & 1{,}5 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 1{,}5 & 0 \\ 1{,}5 & 0 & 1{,}5 \\ 0 & 1{,}5 & 0 \\ 1{,}5 & 3 & 1{,}5 \end{pmatrix}$$

(10)

$$T = \begin{pmatrix} 2 & 4 & 1 \\ 4 & 2 & 5 \\ 3 & 5 & 2 \\ 5 & 6 & 4 \end{pmatrix} \tag{11}$$

3 Set up the linear system of equations corresponding to the discretization shown in figure 2. How many degrees of freedom has the system to be solved?

Using the Principle of Virtual Works for an element subject to body forces:

$$K^e = \int_A B^T DBt\,dA \tag{12}$$

$$f^e = \int_A N^T bt\,dA \tag{13}$$

In this case, K is 6x6 matrix and f a 6x1 vector. The K_{ij} matrices are 2x2 and the f_i vectors are 2x1.

Assembling the matrices:

$$K_g = \begin{pmatrix} k_{33}^1 & k_{31}^1 & 0 & k_{32}^1 & 0 & 0 \\ & k_{11}^1 + k_{22}^2 + k_{33}^3 & k_{31}^3 & k_{12}^1 + k_{21}^2 & k_{23}^2 + k_{32}^3 & 0 \\ & & k_{11}^3 & 0 & k_{12}^3 & 0 \\ & & & k_{22}^1 + k_{11}^2 + k_{33}^4 & k_{13}^2 + k_{31}^4 & k_{32}^4 \\ & symm & & & k_{33}^2 + k_{22}^3 + k_{11}^4 & k_{12}^4 \\ & & & & & k_{22}^4 \end{pmatrix} \tag{14}$$

$$f_g = \begin{pmatrix} f_3^1 + R_1 \\ f_1^1 + f_2^2 + f_3^3 + R_2 \\ f_1^3 + R_3 \\ f_2^1 + f_1^2 + f_3^4 \\ f_3^2 + f_2^3 + f_1^4 + R_5 \\ f_2^4 + R_6 \end{pmatrix} \tag{15}$$

Where R are the reactions:

$$R_1 = \begin{pmatrix} R_{1x} \\ R_{1y} \end{pmatrix} \quad R_2 = \begin{pmatrix} R_{2x} \\ R_{2y} \end{pmatrix} \quad R_3 = \begin{pmatrix} R_{3x} \\ R_{3y} \end{pmatrix} \quad R_5 = \begin{pmatrix} R_{5x} \\ 0 \end{pmatrix} \quad R_6 = \begin{pmatrix} R_{6x} \\ R_{6y} \end{pmatrix} \tag{16}$$

The linear system that we have to solve is:

$$K_g a = f_g \tag{17}$$

4 Compute the FE approximation u^h.

Imposing the boundary conditions shown in (7), (8) and (9) we obtain the following solution (meters):

$$\begin{pmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \\ u_{6x} \\ u_{6y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0{,}00012820 \\ -0{,}0011325 \\ 0 \\ -0{,}0038676 \\ 0 \\ -0{,}01 \end{pmatrix} \tag{18}$$

Once we have the displacements we can calculate the reactions (Newtons):

$$\begin{pmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ R_{3x} \\ R_{3y} \\ R_{5x} \\ R_{6x} \\ R_{6y} \end{pmatrix} = 10^7 \begin{pmatrix} 0{,}11797 \\ 0{,}02674 \\ 0{,}90811 \\ 1{,}13982 \\ -0{,}40287 \\ 2{,}01442 \\ -0{,}05341 \\ -0{,}56980 \\ -3{,}18055 \end{pmatrix} \tag{19}$$