# Homework-1, Finite Elelemt Methods

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Question no-1: Considering following Differential equation:

$$-u^{''} = f \quad \Omega \in [0,1] \tag{1}$$

Given Details:

Boundary Conditions:

$$u(0) = 0 \quad \text{and} \quad u(1) = \alpha \tag{2}$$

Two noded Linear Mesh (Discretization):

$$x_i = ih, \quad \text{and} \quad i = 0, 1, ...., n, \quad h = \frac{1}{n} \tag{3}$$

# 1 Solution

## 1.1 Find the weak form of the problem and write FE approximation of $u^h$

## 1.2 Describing Linear system of Equations

$$-\frac{d^2u}{dx^2} = f \tag{4}$$

Multiplying both sides by some weighting function $w$ and taking integral on both sides over the domain $\Omega$:

$$\int_\Omega -w\frac{d^2u}{dx^2}dx = \int_\Omega wfdx \tag{5}$$

After applying Integartion by parts for the L.H.S of the expression, we get following expression:

$$[w\frac{du}{dx}]_1 - [w\frac{du}{dx}]_0 + \int_0^1 \frac{dw}{dx}\frac{du}{dx}dx = \int_0^1 wfdx \tag{6}$$

$$\int_0^1 \frac{dw}{dx}\frac{du}{dx}dx = \int_0^1 wfdx + [w\frac{du}{dx}]_1 - [w\frac{du}{dx}]_0 \tag{7}$$

Since we are discretizing for $i$ number of nodes, we can formulize the equation (7) in terms of $i = 0, 1, 2, 3...., n$ nodes.

$$\int_0^1 \frac{dw_i}{dx}\frac{du_i}{dx}dx = \int_0^1 w_i f_i dx + [w_i\frac{du}{dx}]_0 - [w_i\frac{du}{dx}]_1 \tag{8}$$

We can use a piecewise linear approximations for $u$ in the following way:

$$u \approx u^h = \sum_{j=0}^{n} N_j(x)u_j \tag{9}$$

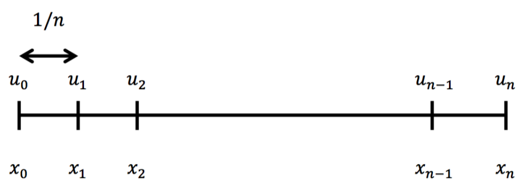


Figure 1: Discretization

### 1.3 1st Choice; Global Definition Of Shape Function

We can propose several choices for Shape function, Using global form of shape function over the domain $\Omega$. Let suppose it's a monomial, Then:

$$u \approx u^h = \sum_{j=0}^{n} x^j \alpha_j = \alpha_0 + \sum_{j=1}^{n} N_j(x)\alpha_j = \alpha_0 + \alpha_1 x + \alpha_2 x^2 ... + \alpha_n x^n \tag{10}$$

Here the Shape function is:

$$N_j(x) = \sum_{j=1}^{n} x^j \quad \text{and} \quad N_j(jh) = (j\frac{1}{n})^j \tag{11}$$

It also satisfies the boundaries:

$$N(0) = 1 \quad N(1) = 1 \tag{12}$$

Approximating also the weighting function $w = N$ with shape function (Using Glarkien approach). Equation 8 can be presented in the following way:

$$\int_0^1 \frac{dN_i}{dx}(\sum_{j=0}^{n} \frac{dN_j(x)}{dx}\alpha_j)dx = \int_0^1 N_i f_i dx + [N_i \frac{du}{dx}]_0 - [N_i \frac{du}{dx}]_1 \tag{13}$$

For total unknowns $i = 1, 2, ..., n$, one can write Equation 13 in terms of Algebraic equations:

$$\int_0^1 \frac{dN_i}{dx}(\sum_{j=1}^{n} \frac{dN_j(x)}{dx}\alpha_j)dx = \int_0^1 N_i f_i dx + [N_i \frac{du}{dx}]_0 - [N_i \frac{du}{dx}]_1 \tag{14}$$

Linear System of Equation can be defined in following way:

$$k_{ij} = \int_0^1 \frac{dN_i}{dx}(\sum_{j=1}^{n} \frac{dN_j(x)}{dx})dx \tag{15}$$

$$f_i = \int_0^1 N_i f_i dx + [N_i \frac{du}{dx}]_0 - [N_i \frac{du}{dx}]_1 \tag{16}$$

Since $[N_i]_0 = 0 and [N_i]_1 = 1$

$$f_i = \int_0^1 N_i f_i dx + [\frac{du}{dx}]_1 = \int_0^1 N_i f_i dx - \bar{q} + q \quad \text{and} \quad \bar{q} = -k[\frac{du}{dx}]_0 \quad \text{and} \quad q = -k[\frac{du}{dx}]_1 \tag{17}$$

Here, if $u$ represents temperature then $k$ is thermal conductivity, $k = 1$

$$[K][u] = [f] \quad \text{and} \quad k_{ij}u_j = f_i \tag{18}$$

### 1.4 2nd Choice: Local Definition Of Global Shape function

The bar is discretized into two-noded 1D finite element. Within each element the unknown function u(x) is approximated using a linear polynomial as:

$$u^e(x) \approx \bar{u}(x) = \sum_{j=0}^{1} N_i \alpha_j = \alpha_0 + \alpha_1 x \tag{19}$$

$$u^e(x) = \frac{x_2^e - x}{x_2^e - x_1^e}u_1^e + \frac{x - x_1^e}{x_2^e - x_1^e}u_2^e = \sum_{1}^{2} N_i^e(x)u_i^e \tag{20}$$

for $i = 1, 2, 3, ..., n$ based on global numbering. One can write the solution of Equation 8 in the following way:

$$\int_0^1 \frac{dN_i^e}{dx}[u_1 \frac{dN_1^e}{dx} + u_2 \frac{dN_2^e}{dx} + ... + u_n \frac{dN_n^e}{dx}]dx = \int_0^1 N_i^e f_i^e dx + [N_i^e \frac{du}{dx}]_0 - [N_i^e \frac{du}{dx}]_1 \tag{21}$$

We can solve the expression for 2 noded linear element (for $i = 1, 2, ..., (n-1)$ elemental division) in the following way: Linear System of Equation can be defined in following way:

$$k_{ij}^e = \int_0^1 \frac{dN_i^e}{dx}[u_1 \frac{dN_1^e}{dx} + u_2 \frac{dN_2^e}{dx}]dx \tag{22}$$

$$f_i^e = \int_0^1 N_i^e f_i dx + [N_i \frac{du}{dx}]_0 - [N_i \frac{du}{dx}]_1 \tag{23}$$

One can write the matrix form of above equation in the following way:

$$[K^e][u] = [f^e] \tag{24}$$

### 1.5 Solving the system for n=3

We will use a 2nd choice (based upon local definition of shape function). we get following discretization $x = [0, 1/3, 2/3, 1]$ and global numbering for nodes $(1, 2, 34)$

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{3} & k_{12}^{3} \\ 0 & 0 & k_{21}^{3} & k_{22}^{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} + \bar{q}_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^2 + f_1^3 \\ f_2^{(3)} - \bar{q}_1 \end{bmatrix} \tag{25}$$

for $i = 1, x_0 = 0$

$$\int_0^1 \frac{dN_1^1}{dx}[u_1 \frac{dN_1^1}{dx} + u_2 \frac{dN_2^1}{dx}]dx = \int_0^1 N_1^1 f_1^1 dx + \bar{q}_0 \tag{26}$$

for $i = 2, x_1 = 1/3$

$$\int_0^{1/3} \frac{dN_2^1}{dx}[u_1 \frac{dN_1^1}{dx} + u_2 \frac{dN_2^1}{dx}]dx + \int_{1/3}^{2/3} \frac{dN_1^2}{dx}[u_2 \frac{dN_1^2}{dx} + u_3 \frac{dN_2^2}{dx}]dx = \int_0^{1/3} N_2^1 f_2^1 dx + \int_{1/3}^{2/3} N_1^2 f_1^2 dx \tag{27}$$

for $i = 3, x_2 = 2/3$

$$\int_{1/3}^{2/3} \frac{dN_2^2}{dx}[u_2 \frac{dN_1^2}{dx} + u_3 \frac{dN_2^2}{dx}]dx + \int_{2/3}^{1} \frac{dN_1^3}{dx}[u_3 \frac{dN_1^3}{dx} + u_4 \frac{dN_2^3}{dx}]dx = \int_{1/3}^{2/3} N_2^2 f_2^2 dx + \int_{2/3}^{1} N_1^3 f_1^3 dx \tag{28}$$

for $i = 4, x_3 = 1$

$$\int_{2/3}^{1} \frac{dN_2^3}{dx}[u_3 \frac{dN_1^3}{dx} + u_4 \frac{dN_2^3}{dx}]dx = \int_{2/3}^{1} N_2^3 f_2^3 dx - \bar{q}_1 \tag{29}$$

After applying local description of shape functions and evaluating integrals. The global matrix equation is therefore written as:

$$\frac{1}{1/3} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} + \bar{q}_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^2 + f_1^3 \\ f_2^{(3)} - \bar{q}_1 \end{bmatrix}$$

Substituting $f = sin(x)$, we get following results: $u_1 = 0$, $u_2 = 0.998$, $u_3 = 1.98$ and $u_4 = 3$. While $\bar{q}_0 = 0.98$ and $\bar{q}_1 = 1$. We get the same results for all values of u from analytical solution. (30)