FEM-homework2

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Part 1

The strong form of the problem is,

$$div\,\sigma = b$$

where σ is the stress second order tensor and b is body force. For a plane stress model, by standard elasticity theory, we have the following relationship between displacement field and strain:

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\gamma_{xz} &= \gamma_{yz} = 0
\end{aligned}$$

where u and v are horizontal and vertical displacement field. The relation between strain and stress in elasticity theory is as follow:

$$\sigma = \mathbf{D}\varepsilon$$

where $\mathbf{D}$ is the elastic material matrix (constitutive matrix),

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

It can be proved from the Maxwell-Betti theorem that the constitutive matrix is always symmetrical. For isotropic elasticity in plane stress model we have,

$$\begin{aligned}
d_{11} &= d_{22} = \frac{E}{1-\nu^2} \\
d_{12} &= d_{21} = \nu d_{11} \\
d_{33} &= \frac{E}{2(1+\nu)}
\end{aligned}$$

where E is the Young Modulus and ν is the Poisson's ratio.

For the boundary condition we should note that at the bottom edge, both horizontal and vertical displacements are 0. In the right edge (y axis), the horizontal displacement is 0 and the vertical displacement of node 6 is equal to -0.01.

Part 2

The nodal coordinate based on the figure in the problem is,

node	x	y
1	-3	0
2	-1.5	0
3	0	0
4	-1.5	1.5
5	0	1.5
6	0	3

And the connectivity matrix is as follow:

$$T = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 2 & 5 \\ 3 & 5 & 2 \\ 5 & 6 & 4 \end{bmatrix}$$

Where the first row is for element 1, the second row is for element 2 and so on and the order of numbering columns shows the order of local numbering for each element.

Part 3

We write nodal displacement for a single element as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

where u_i and v_i are horizontal and vertical displacements respectively and N_i are shape function of node i. The expression for shape functions are,

$$N_i = \frac{1}{2A^{(e)}}(a_i + b_i x + c_i y), \qquad i = 1, 2, 3$$

where,

$$a_i = x_j y_k - x_k y_j, \quad b_i = y_j - y_k, \quad c_i = x_k - x_j, \quad i, j, k = 1, 2, 3$$

and $A^{(e)}$ is area of the element. Note that each N_i is the value 1 in the node i and 0 in the other two nodes. Then the component of strain vector are as follow:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x}u_1 + \frac{\partial N_2}{\partial x}u_2 + \frac{\partial N_3}{\partial x}u_3$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y}v_1 + \frac{\partial N_2}{\partial y}v_2 + \frac{\partial N_3}{\partial y}v_3$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial N_1}{\partial y}u_1 + \frac{\partial N_1}{\partial x}v_1 + \frac{\partial N_2}{\partial y}u_2 + \frac{\partial N_2}{\partial x}v_2 + \frac{\partial N_3}{\partial y}u_3 + \frac{\partial N_3}{\partial x}v_3$$

so, in matrix form we have,

$$\varepsilon = \mathbf{B}\mathbf{a}^{(e)}$$

where ε is the strain vector, $\mathbf{a}^{(e)}$ is displacement vector correspond to the element and $\mathbf{B} = [\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3]$ is the strain matrix that

$$\mathbf{B}_i = \frac{1}{2A^{(e)}} \begin{bmatrix} b_i & 0 \\ 0 & c_i \\ c_i & b_i \end{bmatrix} \qquad i = 1, 2, 3$$

By using constitutive matrix D, the relation between strain and stress is,

$$\sigma = \mathbf{D}\varepsilon = \mathbf{D}\mathbf{B}\mathbf{a}^{(e)}$$

So, by applying PVW to a single element we have,

$$\iint_{A^{(e)}} \mathbf{B}^T \sigma t\, dA - \iint_{A^{(e)}} \mathbf{N}^T \mathbf{b} t\, dA - \oint_{l^{(e)}} \mathbf{N}^T \mathbf{t} t\, ds = \mathbf{q}^{(e)}$$

This equation yields equilibrating nodal forces $\mathbf{q}^{(e)}$ in terms of the nodal forces due to the element deformation (first integral), the body forces (second integral) and the surface traction (third integral). By substituting stress in terms of nodal displacement and by noticing that we do not have any initial stress, initial strain and surface traction, we get

$$\mathbf{K}^{(e)}\mathbf{a}^{(e)} - \mathbf{f}^{(e)} = \mathbf{q}^{(e)}$$

where,

$$\mathbf{K}^{(e)} = \iint_{A^{(e)}} \mathbf{B}^T \mathbf{D} \mathbf{B} t\, dA$$

is the stiffness matrix of the element and

$$\mathbf{f}^{(e)} = \mathbf{f}_b^{(e)} = \iint_{A^{(e)}} \mathbf{N}^T \mathbf{b} t\, dA$$

is the body force (weight). The components of element stiffness matrix takes the form,

$$K_{ij}^{(e)} = \left(\frac{t}{4A}\right)^{(e)} \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix} \tag{1}$$

and in our case, the components of body force are as follow:

$$f_i^{(e)} = f_{b_i}^{(e)} = \frac{At^{(e)}}{3} \begin{bmatrix} b_x \\ b_y \end{bmatrix} \tag{2}$$

where $b_x = 0$ and $b_y = -\rho g$.

The global equilibrium by connectivity matrix in part 2 can be written in matrix form as

$$\mathbf{Ka} = \mathbf{f}$$

where,

$$\mathbf{K} = \begin{bmatrix} K_{33}^{(1)} & K_{31}^{(1)} & 0 & K_{32}^{(1)} & 0 & 0 \\ & K_{11}^{(1)} + K_{22}^{(2)} + K_{33}^{(3)} & K_{31}^{(3)} & K_{12}^{(1)} + K_{21}^{(2)} & K_{23}^{(2)} + K_{32}^{(3)} & 0 \\ & & K_{11}^{(3)} & 0 & K_{12}^{(3)} & 0 \\ & & & K_{22}^{(1)} + K_{11}^{(2)} + K_{33}^{(4)} & K_{13}^{(2)} + K_{31}^{(4)} & K_{32}^{(4)} \\ & & symm & & K_{33}^{(2)} + K_{22}^{(3)} + K_{11}^{(4)} & K_{12}^{(4)} \\ & & & & & K_{22}^{(4)} \end{bmatrix}$$

is the global stiffness matrix and,

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \qquad a_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \quad i = 1, 2, \dots 6$$

and finally,

$$\mathbf{f} = \begin{bmatrix} f_3^{(1)} + r_1 \\ f_1^{(1)} + f_2^{(2)} + f_3^{(3)} + r_2 \\ f_1^{(3)} + r_3 \\ f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ f_3^{(2)} + f_2^{(3)} + f_1^{(4)} + r_{5,x} \\ f_2^{(4)} + r_6 \end{bmatrix}$$

This linear equation system has 12 equations, but by applying boundary conditions, we should just compute v_5, u_4, and v_4 and then we can obtain reaction forces. thus the degree of freedom of this system is 3.

Part 4

Assumptions of the problem are $E = 10GPa$, $\nu = 0.2$, $\delta = 0.01m$, and $\rho g = 10^3 N/m^2$. First of all, we calculate the values of constitutive matrix as we see in part 1, so

$$d_{11} = d_{22} = 10.4167 \qquad d_{12} = d_{21} = 2.0833 \qquad d_{33} = 4.1667$$

Besides, the area of each element is $A = 1.1250$ and the thickness $t = 1$. For element 1 we have,

$$b_1 = 1.5,\ b_2 = 0,\ b_3 = -1.5,\ c_1 = -1.5,\ c_2 = 1.5,\ c_3 = 0$$

for element 2 we have,

$$b_1 = -1.5,\ b_2 = 0,\ b_3 = 1.5,\ c_1 = 1.5,\ c_2 = -1.5,\ c_3 = 0$$

we have the same value for both element 3 and 4 like element 1 because they are the same as element 1 and the way we locally numbered nodes. now we are ready to calculate the components of stiffness matrix by (1) as follow:

$$K_{33}^{(1)} = 0.2222 \begin{bmatrix} 23.4376 & 0 \\ 0 & 9.3751 \end{bmatrix}, \quad K_{31}^{(1)} = 0.2222 \begin{bmatrix} -23.4376 & 4.6874 \\ 9.3751 & -9.3751 \end{bmatrix}$$

$$K_{32}^{(1)} = 0.2222 \begin{bmatrix} 0 & -4.6874 \\ -9.3751 & 0 \end{bmatrix}, \quad K_{11}^{(1)} = 0.2222 \begin{bmatrix} 32.8127 & -14.0625 \\ -14.0625 & 32.8127 \end{bmatrix}$$

$$K_{12}^{(1)} = 0.2222 \begin{bmatrix} -9.3751 & 4.6874 \\ 9.3751 & -23.4376 \end{bmatrix}, \quad K_{22}^{(1)} = 0.2222 \begin{bmatrix} 9.3751 & 0 \\ 0 & 23.4376 \end{bmatrix}$$

$$K_{22}^{(2)} = 0.2222 \begin{bmatrix} 9.3751 & 0 \\ 0 & 23.4376 \end{bmatrix}, \quad K_{21}^{(2)} = 0.2222 \begin{bmatrix} -9.3751 & 9.3751 \\ 4.6874 & -23.4376 \end{bmatrix}$$

$$K_{23}^{(2)} = 0.2222 \begin{bmatrix} 0 & -9.3751 \\ -4.6874 & 0 \end{bmatrix}, \quad K_{11}^{(2)} = 0.2222 \begin{bmatrix} 32.8127 & -14.0625 \\ -14.0625 & 32.8127 \end{bmatrix}$$

$$K_{13}^{(2)} = 0.2222 \begin{bmatrix} -23.4376 & 9.3751 \\ 4.6874 & -9.3751 \end{bmatrix}, \quad K_{33}^{(2)} = 0.2222 \begin{bmatrix} 23.4376 & 0 \\ 0 & 9.3751 \end{bmatrix}$$

$$K_{33}^{(3)} = 0.2222 \begin{bmatrix} 23.4376 & 0 \\ 0 & 9.3751 \end{bmatrix}, \quad K_{31}^{(3)} = 0.2222 \begin{bmatrix} -23.4376 & 4.6874 \\ 9.3751 & -9.3751 \end{bmatrix}$$

$$K_{32}^{(3)} = 0.2222 \begin{bmatrix} 0 & -4.6874 \\ -9.3751 & 0 \end{bmatrix}, \quad K_{11}^{(3)} = 0.2222 \begin{bmatrix} 32.8127 & -14.0625 \\ -14.0625 & 32.8127 \end{bmatrix}$$

$$K_{12}^{(3)} = 0.2222 \begin{bmatrix} -9.3751 & 4.6874 \\ 9.3751 & -23.4376 \end{bmatrix}, \quad K_{22}^{(3)} = 0.2222 \begin{bmatrix} 9.3751 & 0 \\ 0 & 23.4376 \end{bmatrix}$$

$$K_{33}^{(4)} = 0.2222 \begin{bmatrix} 23.4376 & 0 \\ 0 & 9.3751 \end{bmatrix}, \quad K_{31}^{(4)} = 0.2222 \begin{bmatrix} -23.4376 & 4.6874 \\ 9.3751 & -9.3751 \end{bmatrix}$$

$$K_{32}^{(4)} = 0.2222 \begin{bmatrix} 0 & -4.6874 \\ -9.3751 & 0 \end{bmatrix}, \quad K_{11}^{(4)} = 0.2222 \begin{bmatrix} 32.8127 & -14.0625 \\ -14.0625 & 32.8127 \end{bmatrix}$$

$$K_{12}^{(4)} = 0.2222 \begin{bmatrix} -9.3751 & 4.6874 \\ 9.3751 & -23.4376 \end{bmatrix}, \quad K_{22}^{(4)} = 0.2222 \begin{bmatrix} 9.3751 & 0 \\ 0 & 23.4376 \end{bmatrix}$$

Therefore, the stiffness matrix is, (I rounded numbers because of the shortage of space and divided into two part such that the first matrix is the first 10 columns and the second one is the two last columns).

$$\begin{bmatrix} 5.21 & 0 & -5.21 & 1.04 & 0 & 0 & 0 & -1.04 & 0 & 0 \\ 0 & 2.08 & 2.08 & -2.08 & 0 & 0 & -2.08 & 0 & 0 & 0 \\ -5.21 & 2.08 & 14.58 & -3.12 & -5.21 & 1.04 & -4.17 & 3.12 & 0 & -3.12 \\ 1.04 & -2.08 & -3.12 & 14.58 & 2.08 & -2.08 & 3.12 & -10.42 & -3.12 & 0 \\ 0 & 0 & -5.21 & 2.08 & 7.29 & -3.12 & 0 & 0 & -2.08 & 1.04 \\ 0 & 0 & 1.04 & -2.08 & -3.12 & 7.29 & 0 & 0 & 2.08 & -5.21 \\ 0 & -2.08 & -4.17 & 3.12 & 0 & 0 & 14.58 & -3.12 & -10.42 & 3.12 \\ -1.04 & 0 & 3.12 & -10.42 & 0 & 0 & -3.12 & 14.58 & 3.12 & -4.17 \\ 0 & 0 & 0 & -3.12 & -2.08 & 2.08 & -10.42 & 3.12 & 14.58 & -3.12 \\ 0 & 0 & -3.12 & 0 & 1.04 & -5.21 & 3.12 & -4.17 & -3.12 & 14.58 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.08 & -2.08 & 2.08 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.04 & 0 & 1.04 & -5.21 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1.04 \\ -2.08 & 0 \\ -2.08 & 1.04 \\ 2.08 & -5.21 \\ 2.08 & 0 \\ 0 & 5.21 \end{bmatrix}$$

we should note that all components of stiffness matrix multiple by 10^9 since the unit of E that has given is MPa. according to (2) we can get nodal force vector as follow:

$$\mathbf{f} = \begin{bmatrix} 0 \\ -375 \\ 0 \\ -1125 \\ 0 \\ -375 \\ 0 \\ -1125 \\ 0 \\ -1125 \\ 0 \\ -1125 \end{bmatrix}$$

and the displacement vector is,

$$\mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ 0 \\ v_5 \\ 0 \\ -0.01 \end{bmatrix}$$

By solving the 7th, 8th , and 10th equation we get,

$$u_4 = -0.0001m, \qquad v_4 = -0.0011m, \qquad v_5 = -0.0039m$$

and by knowing other displacement values we can find reaction forces as follow:

$$r_1 = \begin{bmatrix} r_{1,x} \\ r_{1,y} \end{bmatrix} = \begin{bmatrix} 1180000 \\ 267000 \end{bmatrix}, \quad r_2 = \begin{bmatrix} r_{2,x} \\ r_{2,y} \end{bmatrix} = \begin{bmatrix} 9081000 \\ 11398000 \end{bmatrix}$$
$$r_3 = \begin{bmatrix} r_{3,x} \\ r_{3,y} \end{bmatrix} = \begin{bmatrix} -4029000 \\ 20144000 \end{bmatrix}, \quad r_6 = \begin{bmatrix} r_{6,x} \\ r_{6,y} \end{bmatrix} = \begin{bmatrix} -5698000 \\ -31806000 \end{bmatrix}$$
$$r_{5,x} = -534000$$