

Homework-1, Finite Element Methods

Mazhar Ali

November 2015

Question no-1: Considering following Differential equation:

$$-u'' = f \quad \Omega \in [0, 1] \quad (1)$$

Given Details:

Boundary Conditions:

$$u(0) = 0 \quad \text{and} \quad u(1) = \alpha \quad (2)$$

Two noded Linear Mesh (Discretization):

$$x_i = ih, \quad \text{and} \quad i = 0, 1, \dots, n, \quad h = \frac{1}{n} \quad (3)$$

1 Solution

1.1 Find the weak form of the problem and write FE approximation of u^h

1.2 Describing Linear system of Equations

$$-\frac{d^2u}{dx^2} = f \quad (4)$$

Multiplying both sides by some weighting function w and taking integral on both sides over the domain Ω :

$$\int_{\Omega} -w \frac{d^2u}{dx^2} dx = \int_{\Omega} w f dx \quad (5)$$

After applying Integration by parts for the L.H.S of the expression, we get following expression:

$$[w \frac{du}{dx}]_0 - [w \frac{du}{dx}]_1 + \int_0^1 \frac{dw}{dx} \frac{du}{dx} dx = \int_0^1 w f dx \quad (6)$$

$$\int_0^1 \frac{dw}{dx} \frac{du}{dx} dx = \int_0^1 w f dx + [w \frac{du}{dx}]_1 - [w \frac{du}{dx}]_0 \quad (7)$$

Since we are discretizing for i number of nodes, we can formulize the equation (7) in terms of $i = 0, 1, 2, 3, \dots, n$ nodes.

$$\int_0^1 \frac{dw_i}{dx} \frac{du_i}{dx} dx = \int_0^1 w_i f_i dx + [w_i \frac{du}{dx}]_0 - [w_i \frac{du}{dx}]_1 \quad (8)$$

We can use a piecewise linear approximations for u in the following way:

$$u \approx u^h = \sum_{j=0}^n N_j(x) u_j \quad (9)$$

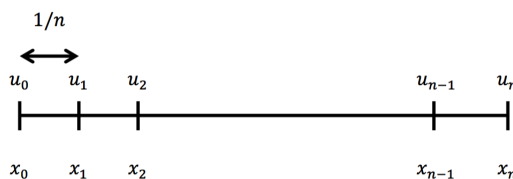


Figure 1: Discretization

1.3 1st Choice; Global Definition Of Shape Function

We can propose several choices for Shape function, Using global form of shape function over the domain Ω . Let suppose it's a monomial, Then:

$$u \approx u^h = \sum_{j=0}^n x^j \alpha_j = \alpha_0 + \sum_{j=1}^n N_j(x) \alpha_j = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \dots + \alpha_n x^n \quad (10)$$

Here the Shape function is:

$$N_j(x) = \sum_{j=1}^n x^j \quad \text{and} \quad N_j(jh) = (j \frac{1}{n})^j \quad (11)$$

It also satisfies the boundaries:

$$N(0) = 1 \quad N(1) = 1 \quad (12)$$

Approximating also the weighting function $w = N$ with shape function (Using Glarkien approach). Equation 8 can be presented in the following way:

$$\int_0^1 \frac{dN_i}{dx} \left(\sum_{j=0}^n \frac{dN_j(x)}{dx} \alpha_j \right) dx = \int_0^1 N_i f_i dx + [N_i \frac{du}{dx}]_0 - [N_i \frac{du}{dx}]_1 \quad (13)$$

For total unknowns $i = 1, 2, \dots, n$, one can write Equation 13 in terms of Algebraic equations:

$$\int_0^1 \frac{dN_i}{dx} \left(\sum_{j=1}^n \frac{dN_j(x)}{dx} \alpha_j \right) dx = \int_0^1 N_i f_i dx + [N_i \frac{du}{dx}]_0 - [N_i \frac{du}{dx}]_1 \quad (14)$$

Linear System of Equation can be defined in following way:

$$k_{ij} = \int_0^1 \frac{dN_i}{dx} \left(\sum_{j=1}^n \frac{dN_j(x)}{dx} \right) dx \quad (15)$$

$$f_i = \int_0^1 N_i f_i dx + [N_i \frac{du}{dx}]_0 - [N_i \frac{du}{dx}]_1 \quad (16)$$

Since $[N_i]_0 = 0$ and $[N_i]_1 = 1$

$$f_i = \int_0^1 N_i f_i dx + [\frac{du}{dx}]_1 = \int_0^1 N_i f_i dx - \bar{q} + q \quad \text{and} \quad \bar{q} = -k [\frac{du}{dx}]_0 \quad \text{and} \quad q = -k [\frac{du}{dx}]_1 \quad (17)$$

Here, if u represents temperature then k is thermal conductivity, $k = 1$

$$[K][u] = [f] \quad \text{and} \quad k_{ij} u_j = f_i \quad (18)$$

1.4 2nd Choice: Local Definition Of Global Shape function

The bar is discretized into two-noded 1D finite element. Within each element the unknown function $u(x)$ is approximated using a linear polynomial as:

$$u^e(x) \approx \bar{u}(x) = \sum_{j=0}^1 N_j \alpha_j = \alpha_0 + \alpha_1 x \quad (19)$$

$$u^e(x) = \frac{x_2^e - x}{x_2^e - x_1^e} u_1^e + \frac{x - x_1^e}{x_2^e - x_1^e} u_2^e = \sum_1^2 N_i^e(x) u_i^e \quad (20)$$

for $i = 1, 2, 3, \dots, n$ based on global numbering. One can write the solution of Equation 8 in the following way:

$$\int_0^1 \frac{dN_i^e}{dx} [u_1 \frac{dN_1^e}{dx} + u_2 \frac{dN_2^e}{dx} + \dots + u_n \frac{dN_n^e}{dx}] dx = \int_0^1 N_i^e f_i^e dx + [N_i^e \frac{du}{dx}]_0 - [N_i^e \frac{du}{dx}]_1 \quad (21)$$

We can solve the expression for 2 noded linear element (for $i = 1, 2, \dots, (n-1)$ elemental division) in the following way: Linear System of Equation can be defined in following way:

$$k_{ij}^e = \int_0^1 \frac{dN_i^e}{dx} [u_1 \frac{dN_1^e}{dx} + u_2 \frac{dN_2^e}{dx}] dx \quad (22)$$

$$f_i^e = \int_0^1 N_i^e f_i dx + [N_i \frac{du}{dx}]_0 - [N_i \frac{du}{dx}]_1 \quad (23)$$

One can write the matrix form of above equation in the following way:

$$[K^e][u] = [f^e] \quad (24)$$

1.5 Solving the system for n=3

We will use a 2nd choice (based upon local definition of shape function). we get following discretization $x = [0, 1/3, 2/3, 1]$ and global numbering for nodes (1, 2, 3, 4)

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} + \bar{q}_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^2 + f_1^3 \\ f_2^{(3)} - \bar{q}_1 \end{bmatrix} \quad (25)$$

for $i = 1, x_0 = 0$

$$\int_0^1 \frac{dN_1^1}{dx} [u_1 \frac{dN_1^1}{dx} + u_2 \frac{dN_2^1}{dx}] dx = \int_0^1 N_1^1 f_1 dx + \bar{q}_0 \quad (26)$$

for $i = 2, x_1 = 1/3$

$$\int_0^{1/3} \frac{dN_2^1}{dx} [u_1 \frac{dN_1^1}{dx} + u_2 \frac{dN_2^1}{dx}] dx + \int_{1/3}^{2/3} \frac{dN_1^2}{dx} [u_2 \frac{dN_1^2}{dx} + u_3 \frac{dN_2^2}{dx}] dx = \int_0^{1/3} N_2^1 f_2 dx + \int_{1/3}^{2/3} N_1^2 f_1 dx \quad (27)$$

for $i = 3, x_2 = 2/3$

$$\int_{1/3}^{2/3} \frac{dN_2^2}{dx} [u_2 \frac{dN_1^2}{dx} + u_3 \frac{dN_2^2}{dx}] dx + \int_{2/3}^1 \frac{dN_1^3}{dx} [u_3 \frac{dN_1^3}{dx} + u_4 \frac{dN_2^3}{dx}] dx = \int_{1/3}^{2/3} N_2^2 f_2 dx + \int_{2/3}^1 N_1^3 f_1 dx \quad (28)$$

for $i = 4, x_3 = 1$

$$\int_{2/3}^1 \frac{dN_2^3}{dx} [u_3 \frac{dN_1^3}{dx} + u_4 \frac{dN_2^3}{dx}] dx = \int_{2/3}^1 N_2^3 f_2 dx - \bar{q}_1 \quad (29)$$

After applying local description of shape functions and evaluating integrals. The global matrix equation is therefore written as:

$$\frac{1}{1/3} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} + \bar{q}_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^2 + f_1^3 \\ f_2^{(3)} - \bar{q}_1 \end{bmatrix}$$

Substituting $f = \sin(x)$, we get following results: $u_1 = 0, u_2 = 1.04, u_3 = 2.058$ and $u_4 = 3$. While $\bar{q}_0 = 3.16$ and $\bar{q}_1 = 2.70$. We get the same results for all values of u from analytical solution. (30)