FEM Homework 1 Artemii Sattarov

Given the differential equation:

$-u'' = f$ in $]0,1[$ with B.C. $u(0)=0$ and $u(1)=\alpha$.

1. Derivation of weak form of the problem and description of FE approximation u^h:

$$-\int_0^1 v_i \frac{du^2}{dx^2}\, dx = \int_0^1 v_i f\, dx$$

integrating LHS by parts:

$$-v_i \frac{du}{dx}\Big|_0^1 + \int_0^1 \frac{du}{dx}\frac{dv_i}{dx}\, dx = \int_0^1 v_i f\, dx$$

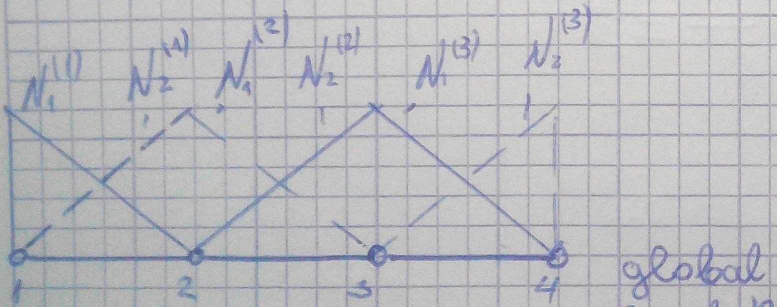
Weak form of the problem ($\frac{du}{dx} = q$):

$$\int_0^1 \frac{du}{dx}\frac{dv_i}{dx}\, dx = \int_0^1 v_i f\, dx + [v_i q]_0 - [v_i q]_e$$

Having chosen Galerkin method, the approximation functions are: $u \simeq u^h = \sum N_i a_i$, $v_i = N_j$

$$\underbrace{\int_0^1 \frac{dN_j}{dx}\frac{d}{dx}\sum N_i}_{\substack{K_{ij}\\(n\times n)}} \cdot \underbrace{a_i}_{\substack{a_i\\(n\times 1)}}\, dx = \underbrace{\int_0^1 N_j f\, dx + [N_j q]_0 - [N_j q]_e}_{\substack{f_j\\(n\times 1)}}$$

2. Description of the linear system of equations



	global	local
$0 \le x \le \frac{1}{3}l$	N_1	$N_1^{(1)}$
$\frac{1}{3}l \le x \le \frac{2}{3}l$	N_1	0
$\frac{2}{3}l \le x \le l$	N_1	0
$0 \le x \le \frac{1}{3}l$	N_2	$N_2^{(1)}$
$\frac{1}{3}l \le x \le \frac{2}{3}l$	N_2	$N_1^{(2)}$
$\frac{2}{3}l \le x \le l$	N_2	0

Node 1: $\int_0^{\frac{1}{3}} \frac{dN_1^{(1)}}{dx}\left(\frac{dN_1^{(1)}}{dx}a_1 + \frac{dN_2^{(1)}}{dx}a_2\right)dx =$

$= \int_0^{\frac{1}{3}} N_1^{(1)} f\, dx + q_0$

	global	local
$0 \le x \le \frac{1}{3}l$	N_3	$N_2^{(2)}$
$\frac{1}{3}l \le x \le \frac{2}{3}l$	N_3	$N_1^{(3)}$
$\frac{2}{3}l \le x \le l$	N_3	
$0 \le x \le \frac{1}{3}l$	N_4	0
$\frac{1}{3}l \le x \le \frac{2}{3}l$	N_4	0
$\frac{2}{3}l \le x \le l$	N_4	$N_2^{(3)}$

Node 2: $\int_0^{\frac{1}{3}l} \frac{dN_2^{(1)}}{dx}\left(\frac{dN_1^{(1)}}{dx}a_1 + \frac{dN_2^{(1)}}{dx}a_2\right)dx +$

$+ \int_{\frac{1}{3}l}^{\frac{2}{3}l} \frac{dN_1^{(2)}}{dx}\left(\frac{dN_1^{(2)}}{dx}a_2 + \frac{dN_2^{(2)}}{dx}a_3\right)dx = \int_0^{\frac{1}{3}l} N_2^{(1)} f\, dx +$

$+ \int_{\frac{1}{3}l}^{\frac{2}{3}l} N_1^{(2)} f\, dx$

Node 3: $\int_{\frac{l}{3}}^{\frac{2l}{3}} \frac{dN_2^{(2)}}{dx}\left(\frac{dN_1^{(2)}}{dx}a_2 + dN_2^{(2)} a_3\right) + \int_{\frac{2}{3}l}^{l} \frac{dN_1^{(3)}}{dx}\left(\frac{dN_1^{(3)}}{dx}a_3 +\right.$

$\left.+ \frac{dN_2^{(3)}}{dx}a_4\right)dx = \int_{\frac{1}{3}l}^{\frac{2}{3}l} N_2^{(2)} f\, dx + \int_{\frac{2}{3}l}^{l} N_1^{(3)} f\, dx$

Node 4: $\int_{\frac{2}{3}l}^{l} \frac{dN_2^{(3)}}{dx}\left(\frac{dN_1^{(3)}}{dx}a_3 + \frac{dN_2^{(3)}}{dx}a_4\right)dx = \int_{\frac{2}{3}l}^{l} \frac{dN_2^{(3)}}{dx} f\, dx - \bar{q}$

The obtained system of linear equations can be represented as:

$$\begin{vmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} \\ 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)} \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{vmatrix} = \begin{vmatrix} f_1^{(1)} + q_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} - \bar{q} \end{vmatrix}$$

3. Computation of FEM approximations:

From the expression of shape function:

$\frac{dN_1^{(e)}}{dx} = -\frac{1}{l^{(e)}} = -3$, $\frac{dN_2^{(e)}}{dx} = \frac{1}{l^{(e)}} = 3$

For 1st element

$K_{11}^{(1)} = K_{11}^{(e)} = \int_0^{\frac{1}{3}} \left(-\frac{1}{l^{(e)}}\right)\left(-\frac{1}{l^{(e)}}\right)dx = 3 = K_{22}^{(e)}$

$K_{12}^{(1)} = K_{12}^{(e)} = \int_0^{\frac{1}{3}} -3 \cdot 3\, dx = -3 = K_{21}^{(1)} = K_{21}^{(e)}$

$f_1^{(1)} = \int_0^{1/3} \frac{x_2^{(e)} - x}{\frac{1}{3}} \sin x\, dx = \int_0^{1/3} 3\left(\frac{1}{3} - x\right)\sin x\, dx = -\cos x\Big|_0^{\frac{1}{3}} -$

$$-3(-x\cos x+\sin x)\Big|_0^{\frac{1}{3}}=0{,}055043-0{,}03663=0{,}01842$$

$$f_2^{(1)}=\int_0^{1/3}\frac{x-x_1^{(1)}}{\frac{1}{3}}\sin x\,dx=\int_0^{1/3}3x\sin x\,dx=(-3x\cos x+3\sin x)\Big|_0^{\frac{1}{3}}=$$

$$=0{,}98158-0{,}94495=0{,}03662$$

For 2nd element:

$$f_1^{(2)}=\int_{1/3}^{2/3}\frac{x_2^{(2)}-x}{\frac{1}{3}}\sin x\,dx=\int_{\frac{1}{3}}^{\frac{2}{3}}(2\sin x-3x\sin x)dx=0{,}07143$$

$$f_2^{(2)}=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{x-x_1^{(2)}}{\frac{1}{3}}\sin x\,dx=0{,}08764$$

For 3rd element

$$f_1^{(3)}=\int_{\frac{2}{3}}^{1}\frac{x_2^{(3)}-x}{\frac{1}{3}}\sin x\,dx=0{,}11659$$

$$f_2^{(3)}=\int_{\frac{2}{3}}^{1}\frac{x-x_1^{(3)}}{\frac{1}{3}}\sin x\,dx=0{,}12899$$

Substituting obtained values into matrix (taking into account $u_1=0$, $u_3=u(1)=3$.

$$\begin{vmatrix}3&-3&0&0\\-3&6&-3&0\\0&-3&6&-3\\0&0&-3&3\end{vmatrix}\begin{vmatrix}0\\u_2\\u_3\\3\end{vmatrix}=\begin{vmatrix}0{,}01842+q_0\\0{,}03662+0{,}07143\\0{,}08764+0{,}11659\\0{,}12899-q\end{vmatrix}$$

$$\begin{cases}6u_2-3u_3=0{,}10805\\-3u_2+6u_3-9=0{,}20423\end{cases}$$

$$9u_3=18{,}51651 \Rightarrow u_3=2{,}05739$$

$u_2=1{,}04670$ Comparison of solutions:

	FEM solution	Analytical
u_1	0	0
u_2	1,04670	1,04670
u_3	2,05739	2,057389
u_4	3	3