AJAY SINGH NEHRA — FEM (HOMEWORK - 1)

Solution:- $-u'' = f$ in $]0,1[$

$\Rightarrow u'' + f(x) = 0$ in $]0,1[$

$u(0) = 0 \qquad u(1) = \alpha$

1. $\int_0^1 w_i\,[u'' + f(x)]\,dx = 0$

$\Rightarrow \int_0^1 w_i\, u''\,dx + \int_0^1 w_i\, f(x)\,dx = 0$

now, integrating by parts

$\int_0^1 w_i u'' = w_i u'\big|_0^1 - \int_0^1 \frac{dw_i}{dx}\left(\frac{du}{dx}\right)dx$

$\Rightarrow w_i u'\big|_0^1 - \int_0^1 \frac{dw_i}{dx}\left(\frac{du}{dx}\right)dx + \int_0^1 w_i f(x)\,dx = 0$

let $R = -\frac{du}{dx}$ (Reaction force on Dirichlet Boundry i.e. at $x=0$)

and $\bar{R} \Rightarrow$ (reaction force at neumann boundry i.e. at $x = l$)

R & $\bar{R}$ is unknown

$$\boxed{\int_0^1 \frac{dw_i}{dx}\frac{du}{dx}\,dx = \int_0^1 w_i f(x)\,dx + [w_i R]_0 - [w_i \bar{R}]_1}$$ weak form

approximation of unknown, we have.

$$u \simeq u^h = \sum_{i=1}^{n} N_i(x)\, u_i$$

$$\int_0^1 \frac{dw_i}{dx}\frac{d}{dx}\left[\sum_{j=1}^{n} N_j(x)\, u_j\right]dx = \int_0^1 w_i f(x)\,dx + [w_i R]_0 - [w_i \bar{R}]_1$$

using Galerkin method

$w_i = N_i$

$$\int_0^1 \frac{dw_i}{dx} \sum_{j=1}^{n} \frac{d}{dx}[N_j(x)]\, \underset{\downarrow\, a}{u_j}\, dx = \int_0^1 N_i f(x)\, dx + \underbrace{[N_i R]_0 - [N_i \bar{R}]_1}_{f_i}$$

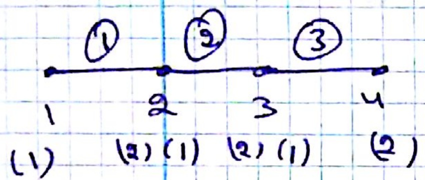
K_{ij}

$$\Rightarrow \quad \underline{\underline{K}}\ \underline{a} = \underline{f}$$

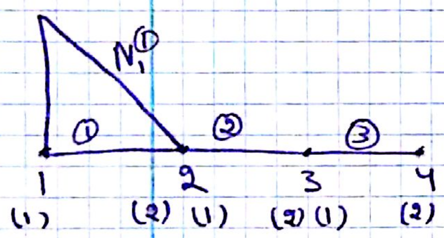
$$K_{ij} = \frac{dN_i}{dx} \frac{d}{dx}[N_j(x)]$$

$$a = u_j$$

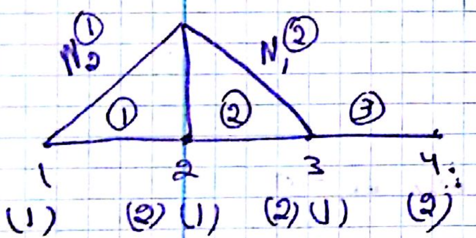
$$f_i = N_i f(x) + [N_i R]_0 - [N_i \bar{R}]_1$$



Shape functions.



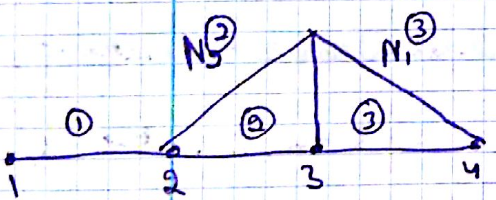
Global		Local	
N_1	=	$N_1^{①}$	$0 \le x \le 1/3$
N_1	=	0	$1/3 \le x \le 2/3$
N_1	=	0	$2/3 \le x \le 1$



$N_2 = N_2^{①} \quad 0 \le x \le 1/3$

$N_2 = N_1^{②} \quad 1/3 \le x \le 2/3$

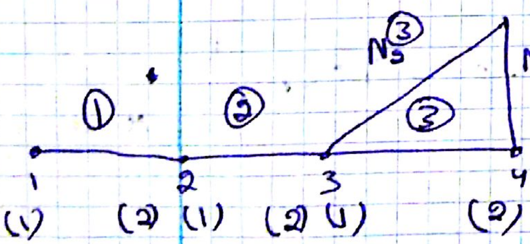
$N_2 = 0 \quad 2/3 \le x \le 1$



$N_3 = 0 \quad 0 \le x \le 1/3$

$N_3 = N_2^{②} \quad 1/3 \le x \le 2/3$

$N_3 = N_1^{③} \quad 2/3 \le x \le 1$



$N_4 = 0 \quad 0 \le x \le 1/3$

$N_4 = 0 \quad 1/3 \le x \le 2/3$

$N_4 = N_2^{③} \quad 2/3 \le x \le 1$

from weak form, we have

$$\int_0^1 \frac{dN_i}{dx}\left[\frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 + \frac{dN_3}{dx}u_3 + \frac{dN_4}{dx}u_4\right] dx = \int_0^1 N_i f(x)\, dx + [N_i R]_0 - [N_i \bar{R}]_1$$

$i = 1, 2, 3, \ldots$

Local system :- (Global solution written in local shape f^n)

for $i=1$

$$\int_0^{1/3} \frac{dN_1^{(1)}}{dx}\left[\frac{dN_1^{(1)}}{dx}u_1 + \frac{dN_2^{(1)}}{dx}u_2\right]dx = \underbrace{\int_0^{1/3} N_1^{(1)} f(x)\, dx}_{f_1^{(1)}} + \underbrace{[N_1 R]_0}_{R} - [N_1 \bar{R}]_1$$

for $i=2$

$$\int_0^{1/3} \frac{dN_2^{(1)}}{dx}\left[\frac{dN_1^{(1)}}{dx}u_1 + \frac{dN_2^{(1)}}{dx}u_2\right]dx + \int_{1/3}^{2/3} \frac{dN_1^{(2)}}{dx}\left[\frac{dN_1^{(2)}}{dx}u_2 + \frac{dN_2^{(2)}}{dx}u_3\right]dx$$

$$= \underbrace{\int_0^{1/3} N_2^{(1)} f(x)\, dx}_{f_2^{(1)}} + 0 + \underbrace{\int_{1/3}^{2/3} N_1^{(2)} f(x)\, dx}_{f_1^{(2)}} + 0$$

for $i=3$

$$\int_{1/3}^{2/3} \frac{dN_2^{(2)}}{dx}\left[\frac{dN_1^{(2)}}{dx}u_2 + \frac{dN_2^{(2)}}{dx}u_3\right]dx + \int_{2/3}^{1} \frac{dN_1^{(3)}}{dx}\left[\frac{dN_1^{(3)}}{dx}u_3 + \frac{dN_2^{(3)}}{dx}u_4\right]dx$$

$$= \underbrace{\int_{1/3}^{2/3} N_2^{(2)} f(x)\, dx}_{f_2^{(2)}} + \underbrace{\int_{2/3}^{1} N_1^{(3)} f(x)\, dx}_{f_1^{(3)}} + 0$$

for $i=4$

$$\int_{2/3}^{1} \frac{dN_2^{(3)}}{dx}\left[\frac{dN_1^{(3)}}{dx}u_3 + \frac{dN_2^{(3)}}{dx}u_4\right]dx = \underbrace{\int_{2/3}^{1} N_2^{(3)} f(x)\, dx}_{f_2^{(3)}} + 0 - [N_3 \underbrace{\bar{R}}_{\bar{R}}]_1$$

In matrix form

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} \\ 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} + R \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} - \bar{R} \end{bmatrix} \quad \text{— (A)}$$

$\therefore$ Stiffness matrix for 'n' 2 noded element

$$K = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & & & & \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & & & \\ & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} & & \\ & & K_{12}^{(3)} & K_{22}^{(3)} + K_{11}^{(4)} & K_{12}^{(4)} & \\ & & & \vdots & & \\ & & & & K_{22}^{n-1} + K_{11}^{n} & K_{12}^{n} \\ & & & & K_{21}^{n} & K_{22}^{n-1} \end{bmatrix}_{n\times n}$$

$$f = \begin{bmatrix} f_1^{(1)} + R \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ \vdots \\ f_2^{(n-2)} + f_1^{(n-1)} \\ f_2^{(n-1)} - R \end{bmatrix}_{n\times 1}, \quad a = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}_{n\times 1}$$

$$u = u^h = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

Now, we have

$$K_{ij} = \int_{\Omega^{e}} \frac{dN_i^{e}}{dx} \frac{dN_j^{e}}{dx}\, dx, \quad f_i = \int_{\Omega^{e}} N_i^{e} f(x)\, dx$$

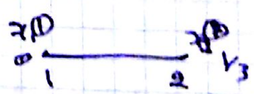
we know that:-

$$u^h(x) = a_0 + a_1(x) \quad \text{[polynomial approximation]}$$

③ finite element approximation u^h for $n=3$; $f(x) = \sin x$; $\alpha = 3$

for $e = 1$, element 1

$$K_{11}^{(1)} = \int_0^{1/3} \left(\frac{-1}{1/3}\right)\left(\frac{-1}{1/3}\right) dx = 3 = K_{22}^{(1)}$$



$$K_{21}^{(1)} = K_{12}^{(1)} = \int_0^{1/3} \left(\frac{-1}{1/3}\right)\left(\frac{1}{1/3}\right) dx = -3$$

$$f_1^{(1)} = \int_0^{1/3} \left(\frac{x_2^{(1)} - x}{1/3}\right) \sin x\, dx = 3 \int_0^{1/3} \left(x_2^{(1)} \sin x - x \sin x\right) dx$$

$$= 3 \int_0^{1/3} \left(\tfrac{1}{3} \sin x - x \sin x\right) dx$$

$$= -\cos x \Big|_0^{1/3} - \Big[3(-x\cos x + \sin x)\Big]_0^{1/3}$$

$$= -\cos\frac{1}{3} + \cos 0 + 3\times\frac{1}{3}\cos\frac{1}{3} - \sin\frac{1}{3} - 3\times 0\times\cos 0 + \sin 0$$

$$= 0.05504 - 3\times 0.01220$$

$$f_1^{(1)} = 0.01844$$

$$f_2^{(1)} = \int_0^{1/3} \left(\frac{x - x_1^{(1)}}{1/3}\right)\sin x\, dx$$

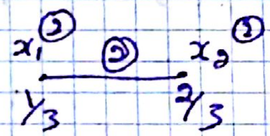
$$= 3\int_0^{1/3} [x\sin x - x_1^{(1)}\sin x]\, dx \qquad (x_1^{(1)} = 0)$$

$$= 3\int_0^{1/3} (x\sin x)\, dx$$

$$= 3\times 0.01220$$

$$f_2^{(1)} = 0.03660$$

$\Rightarrow$ for $e = 2$



$$K_{11}^{(2)} = K_{22}^{(2)} = 3$$

$$K_{12}^{(2)} = K_{21}^{(2)} = -3$$

$$f_1^{(2)} = \int_{1/3}^{2/3} \left(\frac{x_2^{(2)} - x}{1/3}\right)\sin x\, dx$$

$$= 3\int_{1/3}^{2/3} (x_2^{(2)}\sin x - x\sin x)\, dx$$

$$= 3\int_{1/3}^{2/3} \left(\tfrac{2}{3}\sin x - x\sin x\right) dx$$

$$= -\cos x \Big|_{1/3}^{2/3} - 3\Big[-x\cos x + \sin x\Big]_{1/3}^{2/3}$$

$$= -\cos\frac{2}{3} + \cos\frac{1}{3} + 3\times\frac{2}{3}\cos\frac{2}{3} - 3\sin\frac{2}{3} - 3\times\frac{1}{3}\cos\frac{1}{3} + \sin\frac{1}{3}$$

$$= 2\times 0.15907 - 3\,[0.08223]$$

$$\boxed{f_1^{(2)} = 0.07145}$$

$$f_2^{(2)} = \int_{1/3}^{2/3} \left(\frac{x - x_1^{(2)}}{1/3}\right)\sin x\, dx$$

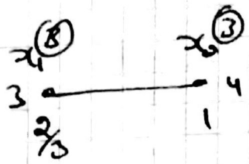
$$= 3\int_{1/3}^{2/3} \left[x \sin x - \frac{1}{3}\sin x\right] dx$$

$$= 3\left[\left[-x\cos x + \sin x\right]_{1/3}^{2/3} + \frac{1}{3}\left[\cos x\right]_{1/3}^{2/3}\right]$$

$$= 3 \times 0.08223 - 0.15907$$

$$\boxed{f_2^{(2)} = 0.08762}$$

for $e = 3$, element ③



$$f_1^{(3)} = \int_{2/3}^{1}\left(\frac{x_2^{(3)} - x}{1/3}\right)\sin x \, dx$$

$$= 3\int_{2/3}^{1}\left(x_2^{(3)}\sin x - x\sin x\right) dx$$

$$= 3\int_{2/3}^{1}\left(\sin x - x\sin x\right) dx$$

$$= 3\int_{2/3}^{1}\sin x \, dx - 3\int_{2/3}^{1} x\sin x \, dx$$

$$= 3\left[-\cos x\right]_{2/3}^{1} - 3\left(-x\cos x + \sin x\right)_{2/3}^{1}$$

$$= 3 \times 0.24558 - 3(0.20672)$$

$$\boxed{f_1^{(3)} = 0.11659}$$

$$f_2^{(3)} = \int_{2/3}^{1}\left(\frac{x - x_1^{(3)}}{1/3}\right)\sin x \, dx$$

$$= 3\int_{2/3}^{1} x\sin x - \cancel{3}\int_{2/3}^{1}\frac{2}{\cancel{3}}\sin x \, dx$$

$$= 3 \times \left[-x\cos x + \sin x\right]_{2/3}^{1} - 2\left[-\cos x\right]_{2/3}^{1}$$

$$= 3 \times 0.20672 - 2(0.24550)$$

$$\boxed{f_2^{(3)} = 0.12899}$$

putting all terms in eq^n (A)

$$3\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix} = \begin{bmatrix}0.01844 + R\\ .03660 + .07145\\ 0.08762 + 0.11659\\ 0.12899 - \bar{R}\end{bmatrix}$$

from data $u(0) = u_1 = 0$
$u(1) = u_4 = \alpha = 3$

so now

$$3\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{bmatrix}0\\ u_2\\ u_3\\ 3\end{bmatrix} = \begin{bmatrix}0.01844 + R\\ 0.03660 + 0.07145\\ 0.08762 + 0.11659\\ 0.12899 - \bar{R}\end{bmatrix}$$

solve for u_1 & u_3

$0 + 6u_2 - 3u_3 = \cancel{0.10805}\ \cancel{0.10805}\ 0.10805$

$0 - 3u_2 + 6u_3 - 9 = 0.20421$

$$\begin{aligned} +6u_2 - 3u_3 &= 0.10805\\ -6u_2 + 12u_3 &= 18.40842\\ \hline 9u_3 &= 18.51647 \end{aligned}$$

$$\boxed{u_3 = 2.05738}$$

$6u_2 - 3(2.05738) = .010805$

$$\boxed{u_2 = 1.03169}$$

~~$u_1 = 0$~~
~~$u_2 = 1.0967$~~
~~$u_3 = 2.05738$~~
~~$u_4 = 3$~~

$$\boxed{\begin{aligned} u_1 &= 0\\ u_2 &= 1.03169\\ u_3 &= 2.05738\\ u_4 &= 3 \end{aligned}} \Rightarrow \text{Fem soluti}$$

R & $\bar{R}$ unknown:

$3(u_1 - u_2) = 0.01844 + R$ $\quad u_1 = 0$

$\Rightarrow -3u_2 = 0.01844 + R$

$-3(1.03169) = 0.01844 + R$

$$\boxed{R = -3.11351}$$

$3(-u_3 + u_4) = 0.12899 - \bar{R}$

$3(-2.05734 + 3) = 0.12899 - \bar{R}$

$\boxed{R = -2.69899}$

Exact solution:-

$u(x) = \sin x + (3 - \sin 1)\, x$

at $x = 0$

$u(0) = \sin 0 + (3 - \sin 1)\, 0$

$\boxed{u(0) = 0}$

at $x = \frac{1}{3}$

$u(\frac{1}{3}) = \sin(\frac{1}{3}) + (3 - \sin 1)\frac{1}{3}$

$= 0.32719 + (3 - 0.8414)\frac{1}{3}$

$\boxed{u(\frac{1}{3}) = 1.04670}$

at $x = \frac{2}{3}$

$u(\frac{2}{3}) = \sin(\frac{2}{3}) + (3 - \sin 1)\frac{2}{3}$

$= 0.61836 + (3 - 0.8414)\frac{2}{3}$

$\boxed{u(\frac{2}{3}) = 2.05738}$

at $x = 1$

$u(1) = \sin(1) + (3 - \sin 1)\, 1$

$= 0.8414 + (3 - 0.8414)$

$\boxed{u(1) = 3}$

FEM Solution.	Exact solution.	error
$u_1 = 0$	$u(0) = 0$	0
$u_2 = 1.03169$	$u(\frac{1}{3}) = 1.04670$	0.01501
$u_3 = 2.05738$	$u(\frac{2}{3}) = 2.05738$	0
$u_4 = 3$	$u(1) = 3$	0