$-u'' = f \quad \text{in }]0,1[\qquad u(0) = 0, \quad u(1) = \alpha$

Constitutive eq. $\rightarrow R = -u'$

R_0 on 0

$\bar{R}$ on 1

$$\Rightarrow \frac{d^2u}{dx^2} + f = 0$$

$$\Rightarrow \int_0^1 \omega_i \frac{d^2u}{dx^2}\,dx + \int_0^1 \omega_i f\,dx = 0$$

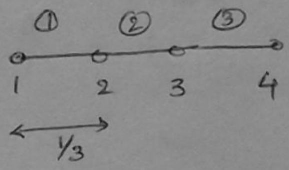
Doing integration by parts

$$\Rightarrow -\int_0^1 \frac{d\omega_i}{dx}\frac{du}{dx}\,dx + \left[\omega_i \frac{du}{dx}\right]_0^1 + \int_0^1 \omega_i f\,dx = 0$$

$$\Rightarrow \left.\begin{aligned}\int_0^1 \frac{d\omega_i}{dx}\frac{du}{dx}\,dx = \int_0^1 \omega_i f\,dx + \left[\omega_i \frac{du}{dx}\right]_0^1\end{aligned}\right\}\text{ Weak form}$$

Let we approximate $u \approx u^h = N_1 u_1 + N_2 u_2 \cdots N_n u_n = \sum_{j=1}^{n} N_j u_j$

$$\Rightarrow \int_0^1 \frac{d\omega_i}{dx}\sum_{j=1}^{n}\frac{dN_j}{dx} u_j\,dx = \int_0^1 \omega_i f\,dx + [\omega_i R]_1 + [\omega_i R]_0$$



Now using Galerkin method

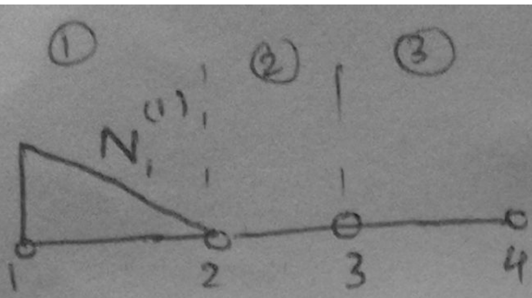
$\omega_i = N_i$; we get

$$\int_0^1 \frac{dN_i}{dx}\left(\sum_{j=1}^{n}\frac{dN_j}{dx} u_j\right)dx = \int_0^1 N_i f\,dx + [N_i R]_1 + [N_i R]_0$$

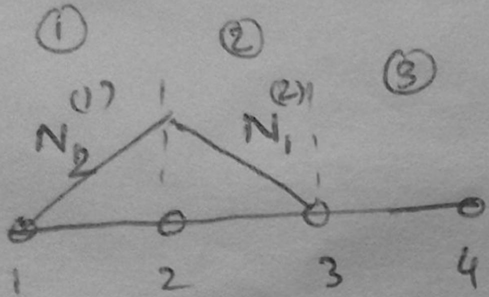
We have to take 4 nodes according to the problem

$$\therefore \int_0^1 \frac{dN_i}{dx}\left(\frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 + \frac{dN_3}{dx}u_3 + \frac{dN_4}{dx}u_4\right)dx = \int_0^1 N_i f\,dx + [N_i R]_1 + [N_i R]_0$$

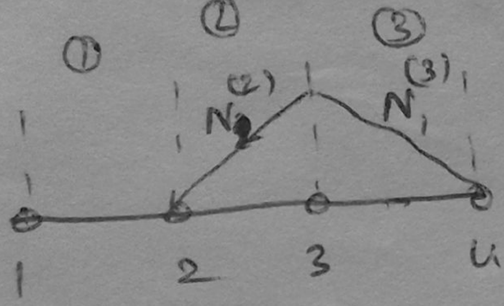
Global solution system



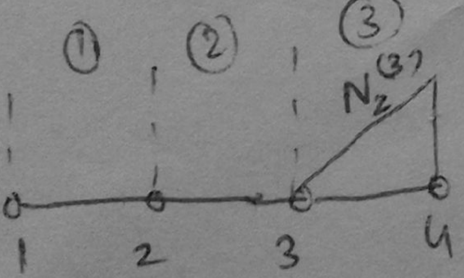
x	Global	Local
$0 \le x \le 1/3$	N_1	$N_1^{(1)}$
$1/3 < x \le 2/3$	N_1	0
$2/3 < x \le 1$	N_1	0



x	Global	Local
$0 \le x \le 1/3$	N_2	$N_2^{(1)}$
$1/3 < x \le 2/3$	N_2	$N_1^{(2)}$
$2/3 < x \le 1$	N_2	0



x	Global	Local
$0 \le x \le 1/3$	N_3	0
$1/3 < x \le 2/3$	N_3	$N_2^{(2)}$
$2/3 < x \le 1$	N_3	$N_1^{(3)}$



x	Global	Local
$0 \le x \le 1/3$	N_4	0
$1/3 < x \le 2/3$	N_4	0
$2/3 < x \le 1$	N_4	$N_2^{(3)}$

For $i=1$

$$\int_0^{1/3} \frac{dN_1^{(1)}}{dx}\left(\frac{dN_1^{(1)}}{dx}u_1 + \frac{dN_2^{(1)}}{dx}u_2\right)dx = \int_0^{1/3} N_1^{(1)} f \, dx + \cancel{\left[N_1^{(1)} R\right]_0} \; R_0$$

For $i=2$

$$\int_0^{1/3} \frac{dN_2^{(1)}}{dx}\left(\frac{dN_1^{(1)}}{dx}u_1 + \frac{dN_2^{(1)}}{dx}u_2\right)dx + \int_{1/3}^{2/3} \frac{dN_1^{(2)}}{dx}\left(\frac{dN_1^{(2)}}{dx}u_1 + \frac{dN_2^{(2)}}{dx}u_2\right)dx = \int_0^{1/3} N_2^{(1)} f\, dx + \int_{1/3}^{2/3} N_1^{(2)} f\, dx$$

For $i=3$

$$\int_{1/3}^{2/3} \frac{dN_2^{(2)}}{dx}\left(\frac{dN_1^{(2)}}{dx}u_1 + \frac{dN_2^{(2)}}{dx}u_2\right)dx + \int_{2/3}^{1} \frac{dN_1^{(3)}}{dx}\left(\frac{dN_1^{(3)}}{dx}u_1 + \frac{dN_2^{(3)}}{dx}u_2\right)dx = \int_{1/3}^{2/3} N_2^{(2)} f\, dx + \int_{2/3}^{1} N_1^{(3)} f\, dx$$

For $i=4$

$$\int_{2/3}^{1} \frac{dN_2^{(3)}}{dx}\left(\frac{dN_1^{(3)}}{dx}u_1 + \frac{dN_2^{(3)}}{dx}u_2\right)dx = \int_{2/3}^{1} N_2^{(3)} f\, dx + \cancel{\left[N_2^{(3)} R\right]_1} \; \bar{R}$$

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The above equations can be written in matrix form

$$
\begin{bmatrix}
k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\
k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\
0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\
0 & 0 & k_{21}^{(3)} & k_{22}^{(3)}
\end{bmatrix}
\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 = \alpha \end{Bmatrix}
=
\begin{Bmatrix} f_1^{(1)} + R_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} + \bar{R} \end{Bmatrix}
$$

$$N_1^e = \frac{x_2 - x}{l^e}, \quad \frac{dN_1^e}{dx} = \frac{-1}{l^e}$$

$$N_2^e = \frac{x - x_1}{l^e} \Rightarrow \frac{dN_2^e}{dx} = \frac{1}{l^e}$$

$$k_{ij} = \int_{l^e} \frac{dN_i}{dx} \cdot \frac{dN_j}{dx} \, dx$$

$$k_{11} = \int_{l^e} \frac{dN_1}{dx} \frac{dN_1}{dx} \, dx = \int_{l^e} \frac{-1}{l^e} \cdot \frac{-1}{l^e} \, dx = \frac{1}{l^e} = \frac{1}{1/3} = 3 = k_{22}$$

$$k_{21} = k_{12} = \int_{l^e} \frac{dN_1}{dx} \cdot \frac{dN_2}{dx} = -3$$

$$N_1^{(1)} = \frac{x_2^{(1)} - x}{l^e}$$

$$N_2^{(1)} = \frac{x - x_1^{(1)}}{1/3} = 3(x-0) = 3x$$

$$= \frac{1/3 - x}{1/3} = 1 - 3x$$

$$f_1^{(1)} = \int_0^{1/3} N_1^{(1)} \sin x\, dx = \int_0^{1/3} \sin x\, dx - 3\int_0^{1/3} x \sin x\, dx$$

$$\int x \sin x$$

$$= \Big[-\cos x\Big]_0^{1/3} - 3\Big[x \int \sin x\, dx\Big]_0^{1/3} + 3\int_0^{1/3}\Big[\Big[\int \sin x\, dx \frac{dx}{dx}\Big]\, dx$$

$$= \Big[-\cos x\Big]_0^{1/3} + 3\Big[x\cos x\Big]_0^{1/3} - 3\int_0^{1/3} \cos x\, dx$$

$$= \Big[-\cos x\Big]_0^{1/3} + 3\Big[x\cos x\Big]_0^{1/3} - 3\Big[\sin x\Big]_0^{1/3}$$

$$= 1 - \cos 1/3 + \frac{3}{3}\cos 1/3 - 3\sin 1/3$$

$$= \cancel{0.9825} = 0.0184$$

$$N_2^{(1)} = \frac{x - x_1^{(1)}}{l^e} = \frac{x - 0}{1/3} = 3x$$

$$f_2^{(1)} = \int_0^{1/3} N_2^{(1)} \sin x\, dx$$

$$= 3\int_0^{1/3} x \sin x\, dx$$

$$= 3\Big[x(-\cos x)\Big]_0^{1/3} - 3\int_0^{1/3}\Big[\frac{dx}{dx}\int \sin x\, dx\Big]\, dx$$

$$= -\frac{3}{3}\cos 1/3 + 3\int_0^{1/3} \cos x\, dx = -\cos 1/3 + 3\Big[\sin x\Big]_0^{1/3}$$

$$= 3\sin 1/3 - \cos 1/3$$

$$= \cancel{0.9825}\ 0.0366$$

$$f_1^{(2)} = \int_{1/3}^{2/3} N_1^{(2)} \sin x \, dx$$

2   3

1/3   2/3

$$N_1^{(2)} = \frac{x_2^{(2)} - x}{l^{(2)}} = \frac{2/3 - x}{1/3} = 2 - 3x$$

$$= \int_{1/3}^{2/3} (2 - 3x) \sin x \, dx$$

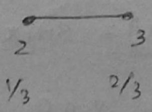
$$= 2\int_{1/3}^{2/3} \sin x \, dx - 3\int_{1/3}^{2/3} x \sin x \, dx$$

$$= -2\left[\cos x\right]_{1/3}^{2/3} + 3\left[x \cos x\right]_{1/3}^{2/3} - 3\left[\sin x\right]_{1/3}^{2/3}$$

$$= 0.3181 + 0.6268 - 0.8735$$

$$= 0.0714$$

$$f_2^{(2)} = \int_{1/3}^{2/3} N_2^{(2)} \sin x \, dx$$



$$N_2^{(2)} = \frac{x - x_1^{(2)}}{1/3} = \frac{x - 1/3}{1/3}$$

$$N_2^{(2)} = 3x - 1$$

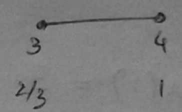
$$= \int_{1/3}^{2/3} 3x \sin x \, dx - \int_{1/3}^{2/3} \sin x \, dx$$

$$= 3\left[\sin x\right]_{1/3}^{2/3} - 3\left[x \cos x\right]_{1/3}^{2/3} + \left[\cos x\right]_{1/3}^{2/3}$$

$$= 0.8735 - 0.6268 - 0.1590$$

$$= 0.0877$$

$$f_1^{(3)} = \int_{2/3}^{1} N_1^{(3)} \sin x \, dx$$



$$N_1^{(3)} = \frac{x_2^{(3)} - x}{1/3}$$

$$N_1^{(3)} = 3 - 3x$$

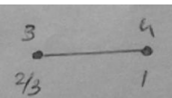
$$= 3\int_{2/3}^{1} \sin x \, dx - 3\int_{2/3}^{1} x \sin x \, dx$$

$$= 3\left[-\cos x\right]_{2/3}^{1} + 3\left[x \cos x\right]_{2/3}^{1} - 3\left[\sin x\right]_{2/3}^{1}$$

$$= 0.7367 + 0.0491 - 0.6693$$

$$= 0.1165$$

$$f_2^{(3)} = \int_{2/3}^{1} N_2^{(3)} \sin x \, dx = \int_{2/3}^{1} 3x \sin x \, dx - \int_{2/3}^{1} 2 \sin x \, dx$$



$$N_2^{(3)} = \frac{x - x_1^{(3)}}{1/3} = \frac{x - 2/3}{1/3} = 3x - 2$$

$$= 2\left[\cos x\right]_{2/3}^{1} - 3\left[x \cos x\right]_{2/3}^{1} + 3\left[\sin x\right]_{2/3}^{1}$$

$$= -0.4911 - 0.0491 + 0.6693$$

$$= 0.1291$$

The matrix now reads as

$$3\begin{bmatrix} \cancel{1} & \cancel{-1} & \cancel{0} & \cancel{0} \\ \cancel{-1} & 2 & -1 & \cancel{0} \\ \cancel{0} & -1 & 2 & \cancel{-1} \\ \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{1} \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{Bmatrix} = \begin{Bmatrix} 0.0184 + R_0 \\ 0.108 \\ 0.2042 \\ 0.1291 + \bar{R} \end{Bmatrix}$$

$$3(2u_2 - u_3) = 0.108 \quad -(1)$$

$$3(-u_2 + 2u_3) - 9 = 0.2042 \quad -(2)$$

$(1) + 2 \times (2)$ gives ~~$3u_3 = 0.5164 \Rightarrow u_3 = 0.1721$~~

$$9u_3 = 18.5164 \Rightarrow u_3 = 2.0573$$

~~$u_2 = 9.2042 - 2u_3$~~

$$6u_2 = 0.108 + 3u_3$$

$$= 6.2799$$

$$\Rightarrow u_2 = 1.04665$$

Exact solution ; $u(x) = \sin x + (3 - \sin 1)x$

$$u(0) = \sin 0 + 0(3 - \sin 1)$$
$$= 0$$

for $x = 1/3$; $u(\frac{1}{3}) = \sin 1/3 + \frac{1}{3}(3 - \sin 1)$

$$= 1.0467$$

for $x = 2/3$, $u(\frac{2}{3}) = \sin \frac{2}{3} + \frac{2}{3}(3 - \sin 1)$

$$= 2.0573$$

for $x = 1$, $u(1) = 3$

FEM Solution	Exact Solution
$u_1 = 0$	$u(0) = 0$
$u_2 = 1.04665$	$u(1/3) = 1.0467$
$u_3 = 2.0573$	$u(2/3) = 2.0573$
$u_4 = 3$	$u(1) = 3$