UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

HOMEWORK ASSIGNMENT 1

Waleed Ahmad Mirza

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Question 1

1) Consider the following differential equation:

$-u'' = f \ in \ ]0.1[$

with the boundary conditions u(0)=0 and u(1)=α.

The finite element discretization is a 2-noded linear mesh given by the nodes $x_i=ih$ for $i=0,1,\ldots,n$ and $h=1/n$.

1) Find the weak form of the problem. Describe the FE approximation u^h.

2) Describe the linear system of equations to be solved.

3) Compute the FE approximation u^h for $n=3$, $f(x) = six\ x$ and $\alpha=3$. Compare it with the exact solution, $u(x)=\sin x + (3-\sin 1)x$.

Solution

$\frac{d^2u}{dx^2} + f = 0$

$\int_0^1 \omega_i \frac{d^2u}{dx^2} dx + \int_0^1 \omega_i f \ dx = 0$

Doing integration by parts

$- \int_0^1 \frac{d\omega_i}{dx} \frac{du}{dx} dx + \left[\omega_i \frac{du}{dx}\right]_0^1 + \int_0^1 \omega_i f \ dx = 0$

$\int_0^1 \frac{d\omega_i}{dx} \frac{du}{dx} dx = \int_0^1 \omega_i f \ dx + \left[\omega_i \frac{du}{dx}\right]_0^1 \quad \Big\} \ Weak \ form$

Let us approximate $u \simeq u^h = N_1 u_1 + N_2 u_2 + \cdots \ N_n u_n = \sum_{j=1}^n N_j u_j$

$\int_0^1 \frac{d\omega_i}{dx} \sum_{j=1}^n \frac{dN_j}{dx} u_j dx = \int_0^1 \omega_i f \ dx + [\omega_i R]_1 + [\omega_i R]_0$

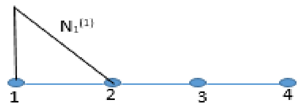
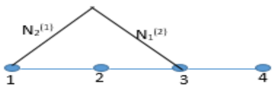
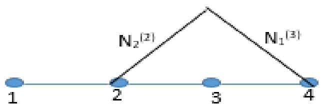
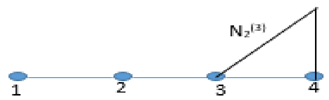
Now using Galerkin method $\omega_i = N_i, we\ get$

$\int_0^1 \frac{dN_i}{dx} (\sum_{j=1}^n \frac{dN_j}{dx} u_j) dx = \int_0^1 N_i f \ dx + [N_i R\ ]_1 + [N_i R]_0$

According to the problem, we have to take four nodes

$\int_0^1 \frac{dN_i}{dx} \left(\frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2 + \frac{dN_3}{dx} u_3 + \frac{dN_4}{dx} u_4\right) dx = \int_0^1 N_i f \ dx + [N_i R]_1 + [N_i R]_0$

<u>Shape functions in element local domain</u>

	X 0≤x≤1/3 1/3<x≤2/3 2/3<x≤1	Local N_1 0 0
	X 0≤x≤1/3 1/3<x≤2/3 2/3<x≤1	Local $N_2^{(1)}$ $N_1^{(2)}$ 0
	X 0≤x≤1/3 1/3<x≤2/3 2/3<x≤1	Local 0 $N_2^{(2)}$ $N_1^{(3)}$
	X 0≤x≤1/3 1/3<x≤2/3 2/3<x≤1	Local 0 0 $N_2^{(3)}$

For i=1,

$$\int_0^{1/3} \frac{dN_1^{(1)}}{dx}\left(\frac{dN_1^{(1)}}{dx}u_1 + \frac{dN_2^{(1)}}{dx}u_2\right)dx = \int_0^{1/3} N_1^{(1)} f\, dx + R_0$$

For i=2,

$$\int_0^{1/3} \frac{dN_2^{(1)}}{dx}\left(\frac{dN_1^{(1)}}{dx}u_1 + \frac{dN_2^{(1)}}{dx}u_2\right)dx + \int_{1/3}^{2/3} \frac{dN_1^{(2)}}{dx}\left(\frac{dN_1^{(2)}}{dx}u_1 + \frac{dN_2^{(2)}}{dx}u_2\right)dx = \int_0^{1/3} N_2^{(1)} f\, dx + \int_{1/3}^{2/3} N_1^{(2)} f\, dx$$

For i=3,

$$\int_{1/3}^{2/3} \frac{dN_2^{(2)}}{dx}\left(\frac{dN_1^{(2)}}{dx}u_1 + \frac{dN_2^{(2)}}{dx}u_2\right)dx + \int_{2/3}^{1} \frac{dN_1^{(3)}}{dx}\left(\frac{dN_1^{(3)}}{dx}u_1 + \frac{dN_2^{(3)}}{dx}u_2\right)dx = \int_{1/3}^{2/3} N_2^{(2)} f\, dx + \int_{2/3}^{1} N_1^{(3)} f\, dx$$

For i=4,

$$\int_{2/3}^{1} \frac{dN_2^{(3)}}{dx}\left(\frac{dN_1^{(3)}}{dx}u_1 + \frac{dN_2^{(3)}}{dx}u_2\right)dx = \int_{2/3}^{1} N_2^{(3)} f\, dx + \bar{R}$$

The above equation can be written in matrix form as follows

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} + R_o \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} + \bar{R} \end{Bmatrix}$$

$$N_1^e = \frac{x_2 - x}{l^e} \ , \frac{dN_1^e}{dx} = -\frac{1}{l^e}$$

$$N_2^e = \frac{x - x_1}{l^e} \ , \frac{dN_2^e}{dx} = \frac{1}{l^e}$$

$$k_{ij} = \int_{l^e} \frac{dN_i}{dx} \cdot \frac{dN_j}{dx} \ dx$$

$$k_{11} = \int_{l^e} \frac{dN_1}{dx} \cdot \frac{dN_1}{dx} dx = \int_{l^e} -\frac{1}{l^e} \cdot -\frac{1}{l^e} dx = -\frac{1}{l^e} = -\frac{1}{\frac{1}{3}} = 3 = k_{22}$$

$$k_{21} = k_{12} = \int_{l^e} \frac{dN_1}{dx} \cdot \frac{dN_1}{dx} dx = -3$$

$$k_{11}^{(1)} = k_{11}^{(2)} = k_{11}^{(3)} = k_{11}$$

$$k_{12}^{(1)} = k_{21}^{(1)} = k_{12}^{(2)} = k_{21}^{(2)} = k_{12}^{(3)} = k_{21}^{(3)} = k_{12}$$

$$k_{22}^{(1)} = k_{22}^{(2)} = k_{22}^{(3)} = k_{22}$$

$$N_1^{(1)} = \frac{x_2^{(1)} - x}{l^e} = \frac{\frac{1}{3} - x}{\frac{1}{3}} = 1 - 3x$$

$$f_1^{(1)} = \int_0^{1/3} N_1^{(1)} sinxdx = \int_0^{\frac{1}{3}} sinxdx - 3\int_0^{1/3} xsinxdx$$

$$f_1^{(1)} = [-cosx]_0^{1/3} - 3[x \int sinxdx]_0^{\frac{1}{3}} + 3\int_0^{1/3} [\int sinxdx]dx$$

$$f_1^{(1)} = [-cosx]_0^{1/3} + 3[x\ cosx]_0^{\frac{1}{3}} - 3\int_0^{1/3} cosxdx$$

$$f_1^{(1)} = [-cosx]_0^{1/3} + 3[x\ cosx]_0^{\frac{1}{3}} - 3[sinx]_0^{\frac{1}{3}}$$

$$f_1^{(1)} = 1 - \cos\frac{1}{3} + \frac{3}{3}\cos\frac{1}{3} - 3sin\frac{1}{3}$$

$$\boldsymbol{f_1^{(1)} = 0.0184}$$

$$N_2^{(1)} = \frac{x - x_1^{(1)}}{l^e} = \frac{x - 0}{\frac{1}{3}} = 3x$$

$$f_2^{(1)} = \int_0^{1/3} N_2^{(1)} sinxdx = 3\int_0^{1/3} xsinxdx$$

$$f_2^{(1)} = 3[x(-cosx)]_0^{1/3} - 3\int_0^{1/3}\left[\frac{dx}{dx}\int sinxdx\right]dx$$

$$f_2^{(1)} = -\frac{3}{3}\cos\frac{1}{3} + 3\int_0^{1/3}\cos x\,dx = -\cos\frac{1}{3} + 3[\sin x]_0^{1/3}$$

$$\boldsymbol{f_2^{(1)} = 0.0366}$$

$$N_1^{(2)} = \frac{x_2^{(2)}-x}{l^e} = \frac{\frac{2}{3}-x}{\frac{1}{3}} = 2-3x$$

$$f_1^{(2)} = \int_{1/3}^{2/3} N_1^{(2)}\sin x\,dx = \int_{1/3}^{2/3}(2-3x)\sin x\,dx = 2\int_{1/3}^{2/3}\sin x\,dx - 3\int_{1/3}^{2/3} x\sin x\,dx$$

$$f_1^{(2)} = -2[\cos x]_{\frac{1}{3}}^{\frac{2}{3}} + 3[xcos\,x]_{\frac{1}{3}}^{\frac{2}{3}} - 3[\sin x]_{\frac{1}{3}}^{\frac{2}{3}}$$

$$\boldsymbol{f_1^{(2)} = 0.0714}$$

$$N_2^{(2)} = \frac{x-x_1^{(2)}}{\frac{1}{3}} = \frac{x-\frac{1}{3}}{\frac{1}{3}} = 3x-1$$

$$f_2^{(2)} = \int_{1/3}^{2/3} N_2^{(2)}\sin x\,dx = \int_{1/3}^{2/3} 3x\sin x\,dx - \int_{1/3}^{2/3}\sin x\,dx$$

$$f_2^{(2)} = 3[\sin x]_{\frac{1}{3}}^{\frac{2}{3}} - 3[xcos\,x]_{\frac{1}{3}}^{\frac{2}{3}} + [\cos x]_{\frac{1}{3}}^{\frac{2}{3}}$$

$$\boldsymbol{f_2^{(2)} = 0.0877}$$

$$N_1^{(3)} = \frac{x_2^{(3)}-x}{\frac{1}{3}} = 3-3x$$

$$f_1^{(3)} = \int_{2/3}^{1} N_1^{(3)}\sin x\,dx = 3\int_{2/3}^{1}\sin x\,dx - 3\int_{2/3}^{1} x\sin x\,dx$$

$$f_1^{(3)} = 3[-\cos x]_{\frac{2}{3}}^{1} + 3[\mathrm{x}\cos x]_{\frac{2}{3}}^{1} - 3[\sin x]_{\frac{2}{3}}^{1}$$

$$\boldsymbol{f_1^{(3)} = 0.1165}$$

$$N_2^{(3)} = \frac{x-x_1^{(3)}}{\frac{1}{3}} = \frac{x-\frac{2}{3}}{\frac{1}{3}} = 3x-2$$

$$f_2^{(3)} = \int_{2/3}^{1} N_2^{(3)}\,sinxdx = \int_{2/3}^{1} 3x\,sinxdx - \int_{2/3}^{1} 2\,sinxdx$$

$$f_2^{(3)} = 2[\cos x]_{\frac{2}{3}}^{1} - 3[xcosx]_{\frac{2}{3}}^{1} + 3[\sin x]_{\frac{2}{3}}^{1} = -0.4911 - 0.0491 + 0.6693$$

$$\boldsymbol{f_2^{(3)} = 0.1291}$$

The matrix now reads as follows:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{Bmatrix} = \begin{Bmatrix} 0.0184 + R_o \\ 0.108 \\ 0.2042 \\ 0.1291 + \bar{R} \end{Bmatrix}$$

Solving above linear equation in MATLAB we get following result.

$u_2 = 1.046704368526853$

$u_3 = 2.057389146531140$

Discussion

In this assignment finite element method has been used to solve the given problem. And values of $\boldsymbol{u}$ obtained as a result is approximated answer and should be verified against analytical solution to establish accuracy of this result.

$u(x) = \sin x + (3 - \sin 1)x$

For x = 0

$u(0) = \sin 0 + 0(3 - \sin 1) = 0$

For $x = \frac{1}{3}$

$u\left(\frac{1}{3}\right) = \sin\frac{1}{3} + \frac{1}{3}(3 - \sin 1) = 1.046704368526853$

For $x = \frac{2}{3}$

$u\left(\frac{2}{3}\right) = \sin\frac{2}{3} + \frac{2}{3}(3 - \sin 1) = 2.057389146531139$

For $x = 1$

$u(1) = 3$

Table 1 Comparison of FEM and Analytical Results

FEM solution	**Exact solution**	**Relative percentage error (%)**
$u_{(0)}$=0	$u_{(0)}$=0	Boundary condition
$u_{(1/3)}$= 1.046704368526853	$u_{(1/3)}$= 1.046704368526853	≈ 0 %
$u_{(2/3)}$= 2.057389146531140	$u_{(2/3)}$= 2.057389146531139	≈ 0 %
$u_{(1)}$=3	$u_{(1)}$=3.0	Boundary condition

Furthermore, a MATLAB code is formulated to solve the given problem using arbitrary number of elements. For four elements, result obtained is presented in Table 2.

Table 2 Comparison of FEM and Analytical Results for four elements

FEM solution from MATLAB	**Exact solution**	**Relative percentage error (%)**
$u_{(0)}$= 0	$u_{(0)}$=0	Boundary condition
$u_{(0.25)}$= 0.787036213052549	$u_{(0.25)}$= 0.787036213052549	≈ 0 %
$u_{(0.50)}$= 1.558690046200255	$u_{(0.50)}$= 1.558690046200255	≈ 0 %
$u_{(0.75)}$= 2.300535521417412	$u_{(0.75)}$= 2.300535521417412	≈ 0 %
$u_{(1)}$=3	$u_{(1)}$=3	Boundary condition

References

1) Eugenio Oñate, Pedro Díez, Francisco Zárate & Antonia Larese (2008). *Introduction to the Finite Element Method*.