homework1-FEM

Mohsen Abedin Nejad

November 2015

Problem

consider the following differential equation

$$-u'' = f \; in \;]0,1[$$

with the boundary conditions $u(0) = 0$ and $u(1) = \alpha$.

1. Find the weak form of the problem. Describe the FE approximation u^h.
2. Describe the linear systems of equations to be solved.
3. Compute the FE approximation u^h for $n = 3$, $f(x) = \sin x$ and $\alpha = 3$ and $\alpha = 3$. Compare it with the exact solution, $u(x) = \sin x + (3 - \sin 1)x$.

Solution

To find the weak form of the problem we should multiple the equation by an arbitrary test function and integrate it. So,

$$-\int_0^1 W(x)u''(x)\,dx = \int_0^1 W(x)f(x)\,dx$$

By integration by part, we have

$$\int_0^1 W'(x)u'(x)dx = \int_0^1 W(x)f(x)\,dx + W(x)u'(x)\Big|_0^1 \tag{1}$$

Equation (1) is the weak form of our differential equation. To describe the FE approximation of u^h we assume that $u \approx u^h = \sum_{j=1}^{n} u_j N_j(x)$, where we choose N_j's as linear functions which have compact support. u_j's are our nodes and are the value of function u in some points of the domain and are unknown except u_1 and u_n by boundary conditions. by assumption that $u'(1) = B$ and $u'(0) = A$, we can write (1) as follow:

$$\int_0^1 W'(x) \sum_{j=1}^{n} u_j \frac{dN_j}{dx}\,dx = \int_0^1 W(x)f(x)\,dx + W(1)B - W(0)A \tag{2}$$

The equation (2) is one equation with n unknowns. since W is an arbitrary function, we can choose different W's to form n equations. In finite element method we use Galerkin Method where $W_i(x) = N_i(x)$ for $i = 1, 2, \ldots, n$. so we write (2), as follow:

$$\int_0^1 \frac{dN_i}{dx} \sum_{j=1}^{n} u_j \frac{dN_j}{dx}\, dx = \int_0^1 N_i(x) f(x)\, dx + N_i(1)B - N_i(0)A \tag{3}$$

for i=1,2,...,n. So the matrix form of (3) is,

$$\begin{bmatrix} K_{11} & K_{12} & \ldots & K_{1n} \\ K_{21} & K_{22} & \ldots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \ldots & K_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \tag{4}$$

where,

$$K_{ij} = \int_0^1 \frac{dN_i}{dx} \frac{dN_j}{dx}\, dx$$
$$f_i = \int_0^1 N_i(x) f(x)\, dx + N_i(1)B - N_i(0)A$$

The equations (3) and (4) are the global form, but to compute we need the local form.

As we have n nodes, there are $n-1$ elements (the distance between two continuous nodes) which their length are $1/(n-1)$. since each N_j is equal to 0 in each element except in j'th and $j-1$'th elements, and also that in these two elements, N_j's are equal to one of N_1 or N_2, therefore we can write (3) as following:

for $i = 1$

$$\int_0^{1/(n-1)} \frac{dN_1^{(1)}}{dx}\Big(u_1 \frac{dN_1^{(1)}}{dx} + u_2 \frac{dN_2^{(1)}}{dx}\Big)\, dx = \int_0^{1/(n-1)} N_1^{(1)} f\, dx - A$$

for $i = 2$

$$\begin{aligned} &\int_0^{1/(n-1)} \frac{dN_2^{(1)}}{dx}\Big(u_1 \frac{dN_1^{(1)}}{dx} + u_2 \frac{dN_2^{(1)}}{dx}\Big)\, dx \\ &+ \int_{1/(n-1)}^{2/(n-1)} \frac{dN_1^{(2)}}{dx}\Big(u_2 \frac{dN_1^{(2)}}{dx} + u_3 \frac{dN_2^{(2)}}{dx}\Big)\, dx \\ &= \int_0^{1/(n-1)} N_2^{(1)} f\, dx + \int_{1/(n-1)}^{2/(n-1)} N_1^{(2)} f\, dx \\ &\vdots \end{aligned}$$

and finally for $i = n$

$$\int_{(n-2)/(n-1)}^{1} \frac{dN_2^{(n-1)}}{dx}\Big(u_{n-1}\frac{dN_1^{(n-1)}}{dx} + u_n\frac{dN_2^{(n-1)}}{dx}\Big)\,dx = \int_{(n-2)/(n-1)}^{1} N_2^{(n-1)} f\,dx + B$$

and we can write the matrix equation (4) as following:

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & \cdots & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 & \cdots & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & K_{21}^{(n-2)} & K_{22}^{(n-2)} + K_{11}^{(n-1)} & K_{12}^{(n-1)} \\ 0 & \cdots & 0 & 0 & K_{21}^{(n-1)} & K_{22}^{(n-1)} \end{bmatrix}$$

$$\times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} f_1^{(1)} - A \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ \vdots \\ f_2^{(n-2)} + f_1^{(n-1)} \\ f_2^{(n-1)} + B \end{bmatrix}$$

In this matrix equation, coefficient matrix K is known, and by boundary conditions the values of $u_1 = 0$ and $u_n = \alpha$. $f_1^{(e)}$ and $f_2^{(e)}$ are also known. So we have n unknowns which are $u_2, \ldots, u_{n-1}, A, B$ and n equations.

To solve our problem with 4 nodes in the domain, first we need to get $N_1^{(e)}$ and $N_2^{(e)}$,

$$N_1^{(e)} = \frac{x_2^{(e)} - x}{l^{(e)}}, \quad N_2^{(e)} = \frac{x - x_1^{(e)}}{l^{(e)}}$$

where $l^{(e)}$ is the length of each element. so we have,

$$N_1^{(1)} = 3\big(\frac{1}{3} - x\big), \quad N_2^{(1)} = 3x$$
$$N_1^{(2)} = 3\big(\frac{2}{3} - x\big), \quad N_2^{(2)} = 3\big(x - \frac{1}{3}\big)$$
$$N_1^{(3)} = 3(1 - x), \quad N_2^{(3)} = 3\big(x - \frac{2}{3}\big)$$

so we can compute coefficient matrix as following:

$$K_{11}^{(1)} = \int_0^{1/3} 9\,dx = 3, \quad K_{12}^{(1)} = \int_0^{1/3} -9\,dx = -3$$
$$K_{21}^{(1)} = \int_0^{1/3} -9\,dx = -3, \quad K_{22}^{(1)} = \int_0^{1/3} 9\,dx = 3$$
$$K_{11}^{(2)} = \int_{1/3}^{2/3} 9\,dx = 3, \quad K_{12}^{(2)} = \int_{1/3}^{2/3} -9\,dx = -3$$
$$K_{21}^{(2)} = \int_{1/3}^{2/3} -9\,dx = -3, \quad K_{22}^{(2)} = \int_{1/3}^{2/3} 9\,dx = 3$$
$$K_{11}^{(3)} = \int_{2/3}^{1} 9\,dx = 3, \quad K_{12}^{(3)} = \int_{2/3}^{1} -9\,dx = -3$$
$$K_{21}^{(3)} = \int_{2/3}^{1} -9\,dx = -3, \quad K_{22}^{(3)} = \int_{2/3}^{1} 9\,dx = 3$$

then,

$$f_1^{(1)} = \int_0^{1/3} 3\big(\frac{1}{3} - x\big)\sin x\,dx = -\cos x\Big|_0^{1/3} + 3x\cos x\Big|_0^{1/3} - 3\sin x\Big|_0^{1/3}$$
$$= 1 - 3\sin\frac{1}{3} \simeq 0.0184$$

$$f_2^{(1)} = \int_0^{1/3} 3x\sin x\,dx = -3x\cos x\Big|_0^{1/3} + 3\sin x\Big|_0^{1/3}$$
$$= -\cos\frac{1}{3} + 3\sin\frac{1}{3} \simeq 0.0366$$

$$f_1^{(2)} = \int_{1/3}^{2/3} 3\big(\frac{2}{3} - x\big)\sin x\,dx = -2\cos x\Big|_{1/3}^{2/3} + 3x\cos x\Big|_{1/3}^{2/3} - 3\sin x\Big|_{1/3}^{2/3}$$
$$= \cos\frac{1}{3} - 3\sin\frac{2}{3} + 3\sin\frac{1}{3} \simeq 0.0714$$

$$f_2^{(2)} = \int_{1/3}^{2/3} 3\big(x - \frac{1}{3}\big)\sin x\,dx = -3x\cos x\Big|_{1/3}^{2/3} + 3\sin x\Big|_{1/3}^{2/3} + \cos x\Big|_{1/3}^{2/3}$$
$$= -\cos\frac{2}{3} + 3\sin\frac{2}{3} - 3\sin\frac{1}{3} \simeq 0.0876$$

$$f_1^{(3)} = \int_{2/3}^{1} 3(1 - x)\sin x\,dx = -3\cos x\Big|_{2/3}^{1} + 3x\cos x\Big|_{2/3}^{1} - 3\sin x\Big|_{2/3}^{1}$$
$$= \cos\frac{2}{3} - 3\sin 1 + 3\sin\frac{2}{3} \simeq 0.1166$$

$$
\begin{aligned}
f_2^{(3)} &= \int_{2/3}^{1} 3\big(x-\frac{2}{3}\big)\sin x\,dx = -3x\cos x\Big|_{2/3}^{1} + 3\sin x\Big|_{2/3}^{1} + 2\cos x\Big|_{2/3}^{1} \\
&= -\cos 1 + 3\sin 1 - 3\sin\frac{2}{3} \simeq 0.1290
\end{aligned}
$$

by assuming that $\alpha = 3$, the linear systems to be solved are:

$$
\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}
\begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{bmatrix}
=
\begin{bmatrix} 0.0184 - A \\ 0.1080 \\ 0.2042 \\ 0.1290 + B \end{bmatrix}
\tag{5}
$$

So by (5), we have,

$$
\begin{aligned}
&6u_2 - 3u_3 = 0.1080, \quad -3u_2 + 6u_3 - 9 = 0.2042 \\
&\Longrightarrow u_2 = 1.0467, \quad u_3 = 2.0574
\end{aligned}
$$

and $B = 2.6988$, $A = 3.1585$. Therefore, we have

$$
\begin{aligned}
u^h &= \sum_{j=1}^{4} u_j N_j(x) = u_1 N_1(x) + u_2 N_2(x) + u_3 N_3(x) u_4 N_4(x) \\
&= \begin{cases} u_1 N_1^{(1)} + u_2 N_2^{(1)} & \text{If } u_1 \le x < u_2 \\ u_2 N_1^{(2)} + u_3 N_2^{(2)} & \text{If } u_2 \le x < u_3 \\ u_3 N_1^{(3)} + u_4 N_2^{(3)} & \text{If } u_3 \le x < u_4 \end{cases} \\
&= \begin{cases} 1.0467(3x) & \text{If } 0 \le x < 1/3 \\ 1.0467(2-3x) + 2.0574(3x-1) & \text{If } 1/3 \le x < 2/3 \\ 2.0574(3-3x) + 3(3x-2) & \text{If } 2/3 \le x \le 1 \end{cases}
\end{aligned}
$$

The values of u_2 and u_3 are axactly the same as the exact solution which is $u(x) = \sin x + (3 - \sin 1)x$ and the u^h is a good linear approximation of this nonlinear function.