Computational mechanics tools Assignment 1: PDE-Toolbox

13th December

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Objective

The main purpose of this assignment is to solve the following problem with Matlab PDE – Toolbox. In order to do so, first it will be considered the mentioned below.

A partial differential equation of the form:

$$u_t - \Delta u = f \quad in \quad \Omega = [0,1]^2 \quad f(x,y,t) = -3e^{-3t}.$$

The initial condition to be considered is:

$$u(x,y,t=0) = x^2 + xy - y^2 + 1$$

And the boundary conditions:

$$\begin{aligned} u_n(x=0,y,t) &= -y, \\ u_n(x=1,y,t) &= 2+y, \\ u(x,y=0,t) &= x^2 + e^{-3t}, \\ u_n(x,y=1,t) &= x-2, \end{aligned}$$

Where:

$$u_n \equiv \partial u / \partial n$$

The analytical solution of the problem is:

$$u(x,y,t) = x^2 + xy - y^2 + e^{-3t}.$$

Method

The used method consists on creating the working domain. The case requires to be a squared domain of $[0,1]^2$. Once the geometry is built, the setting of the boundary conditions goes next. Then, the PDE form needs to be specified. As it is forementioned, the problem considers a parabolic second order differential equation in function of x,y,t. As the form of the parabolic equation follows the form of

$$d\frac{\partial u}{\partial t} - \nabla \cdot (c\nabla u) + au = f$$

then the values for d,c,a,f need to be specified as shown:

- d=1.00,
- c=1.00,
- a=0.0,
- f=-3*exp(-3*t).

Moreover, the solution requires initial conditions such as the one shown before and a time interval due to its time dependency.

The problem states the following topics:

- Theoretical convergence order for T_{end}=10 with an initial mesh refined up to 4 times.
- Theoretical convergence order for T_{end}=1 with an initial mesh refined up to 4 times.
- Finding an efficient way of solving T_{end}=50, knowing by beforehand that $e^{-50} \approx 0$.

Results and conclusions

The considered working domain, once has been solved the partial differential equation shows:

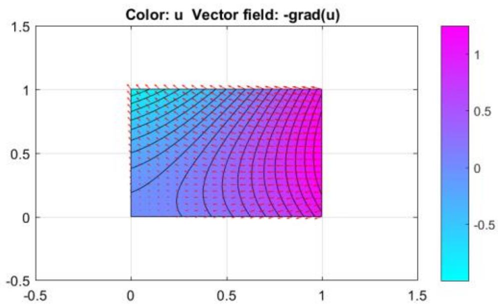

Figure 1. Post-processing for PDE

The number of elements for the meshes 1(being the initial) to 5 (being the last refined mesh) are:

- Mesh 1: 328
- Mesh 2: 1312
- Mesh 3: 5248
- Mesh 4: 20992
- Mesh 5: 83968

H=sqrt(2/N), where N is the number of elements.

Mesh	H (element size)	Error (t=50, Elliptic)	Error (t=50)	Error (t=10)	Error (t=1)
1	0.078086880944303	0.006662018152480	0.006662018152397	0.006662135271409	0.007388099174315
2	0.039043440472152	0.001973319082695	0.001973319082656	0.001973332296640	0.002464530877646
3	0.019521720236076	0.000567875724581	0.000567875724570	0.000567874536164	0.001016772029227
4	0.009760860118038	0.000159921210043	0.000159921210014	0.000160036444225	0.000617278255183
5	0.004880430059019	0.000044413657127	0.000044413657064	0.000044423434873	0.000506165059379

Chart 1. Error values for cases t=50 (elliptic), t=50, t=10, t=1, in function of element size for meshes from 1 to 5.

Slopes	(t=50, Elliptic)	(t=50)	(t=10)	(t=1)
1-2	1.755335028514127	1.755335028524667	1.755350730318127	1.583890176121140

2-3	1.796977110316675	1.796977110316367	1.796989790200354	1.277316808030096
3-4	1.828213947707774	1.828213947941978	1.827171742310876	0.720003372694108
4-5	1.848286017099379	1.848286018896197	1.849007626364158	0.286313049027545

Chart 2. Slope for the cases t=50 (elliptic pde), t=50, t=10, t=1.

As it can be seen for the order of convergence of the slope, h=2, the numerical method used to solve the partial differential equation is of order 2. This gives a quadratic convergence as it is shown in the following plots. It is noticeable stability for values of t=10, t=50 but instability in the order of convergence when referring to t=1. [Be noticed that values on x axis as well as in y axis are in terms of logarithmic scale]

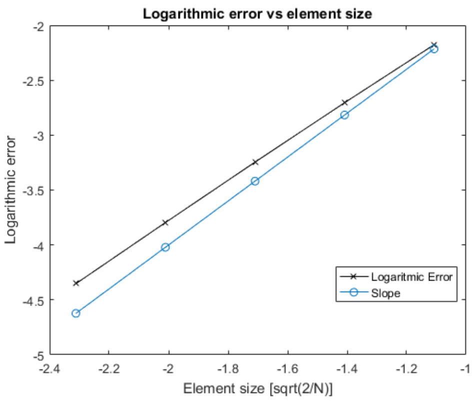

Figure 2. Log error and theoretical slope (h=2). T=10.

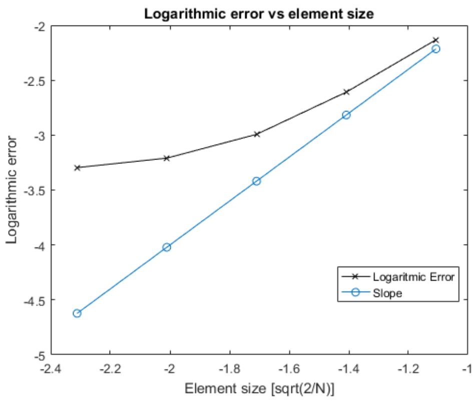

Figure 3. Log error and theoretical slope (h=2). T=1.

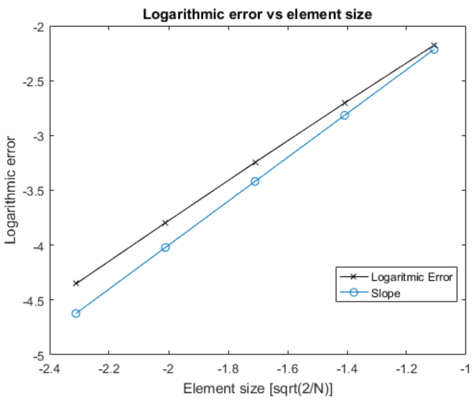

Figure 4. Log error and theoretical slope (h=2). T=50.

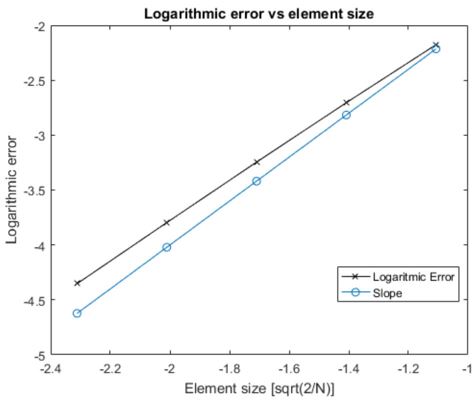

Figure 5. Log error and theoretical slope (h=2). T=50(elliptic).

So as to answer the sentences in this problem, the conclusions will be worked out in the same order as they have been stated.

- T_{end}=10

For t=10, the method reaches good convergence error as the Figure 2 says so. As long as the mesh gets more refined, the slope of the obtained results gets closer to the theoretical slope (ideal one).

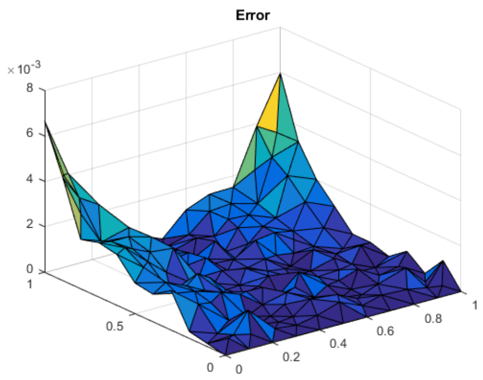

Figure 6. Absolute error between analytical and numerical solutions (t=10, mesh 1).

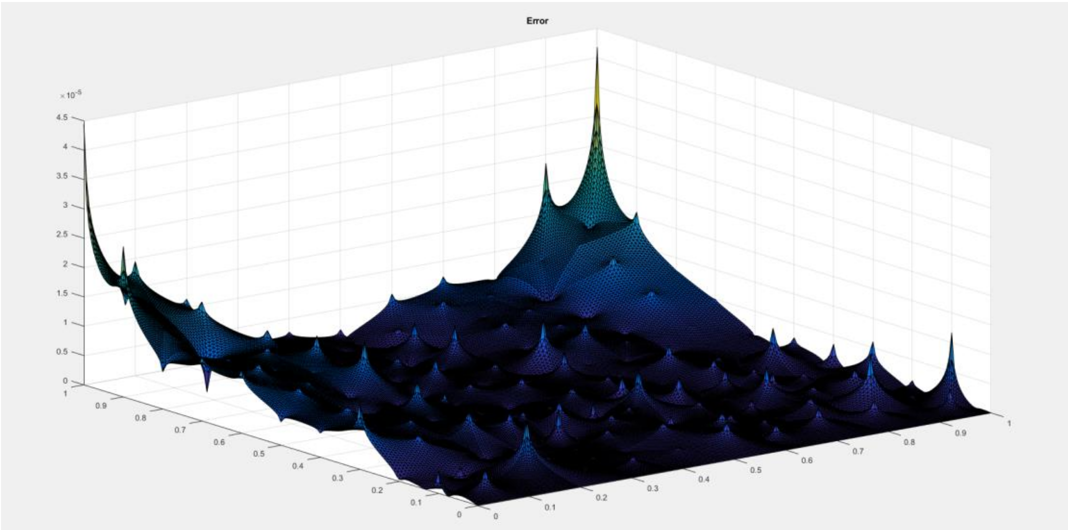

Figure 7. Absolute error between analytical and numerical solutions (t=10, mesh 5).

As shown in these plots, the error decreases as the mesh has more number of elements. Comparatively, it may also be visible the zone where the error still remains high. This is near to the boundaries where there is a change between different boundary conditions. This means that, the solution has to match with the specification of the different boundary conditions but taking into account that on the corners there is a confluence between both of them.

As the maximum error tends to 0, it is understandable that the numerical solution is in a good situation and there are no problems in numerical solution or in the procedure of solving the pde.

To have a better insight we see the consistency between analytical solution and the numerical one.

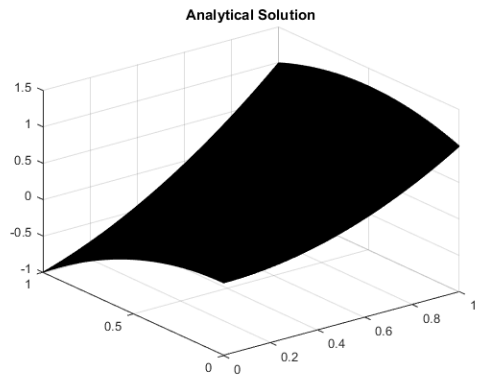

Figure 8.Analytical solution (t=10, mesh 5).

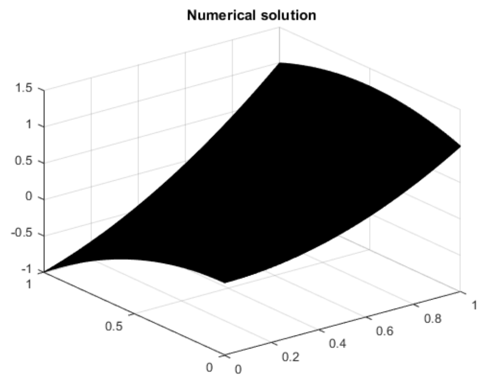

Figure 9. Numerical solution (t=10, mesh 5).

Part 2 states the effect the time makes on the solution when the final time gets modified. It might be seen that because of the source term appearing in the form of exp(-3*t), as long as the time increases, the exponential term becomes smaller. It is also stated in the assignment form that exp(-50) is close to 0 so then this source term could be negligible. This may mean that for t=16.6 since exp(-3*16.6667) is exp(-50) approximately. But this will be explained later on.

For T=1, it is clearly seen in Figure 3 and even in the Chart 1, that it exists a time dependency when the value of the exponential is small enough. This means that this term greatly affects the behaviour in the results of the partial differential equation. The slope of the convergence order shows instability when interpolating between refined meshes. This could be the result of working with high order time terms in the exponential source terms. It should be advisable more refinement so as to get the order of convergence.

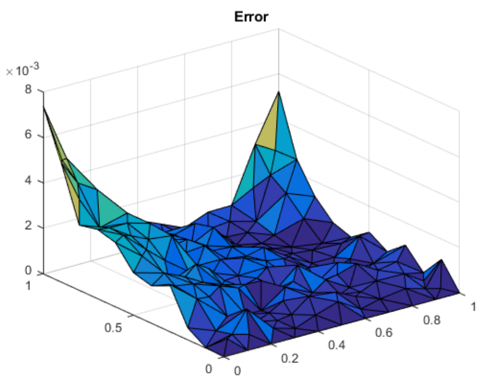

Figure 10. Abs. error analytical and numerical solutions (t=1, mesh 1).

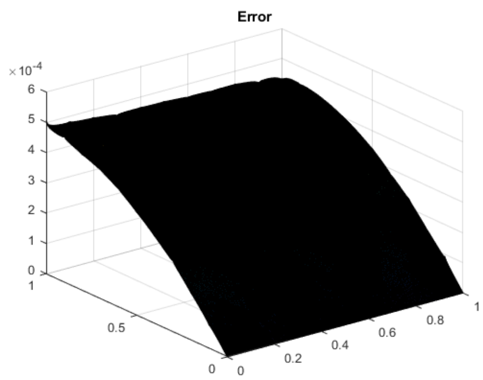

Figure 11 Abs. error analytical and numerical solutions (t=10, mesh 5).

For t=50 it is shown in Figure 4 that the method has reached stability and the convergence order holds. It could be appreciated in Chart 1 the little difference between the solutions for t=10 and t=50. This means that the error convergence holds even after than t=10 probably.

Plots shown below reflect the same behaviour as in t=10.

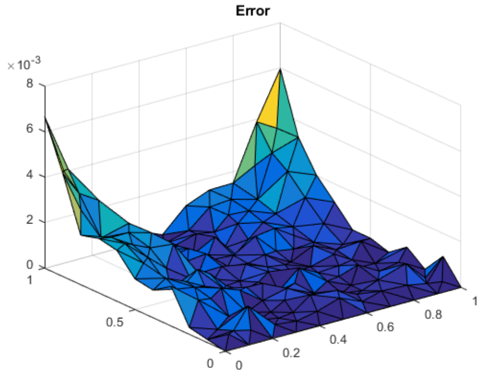

Figure 12. Absolute error between analytical and numerical solutions (t=50, mesh 1).

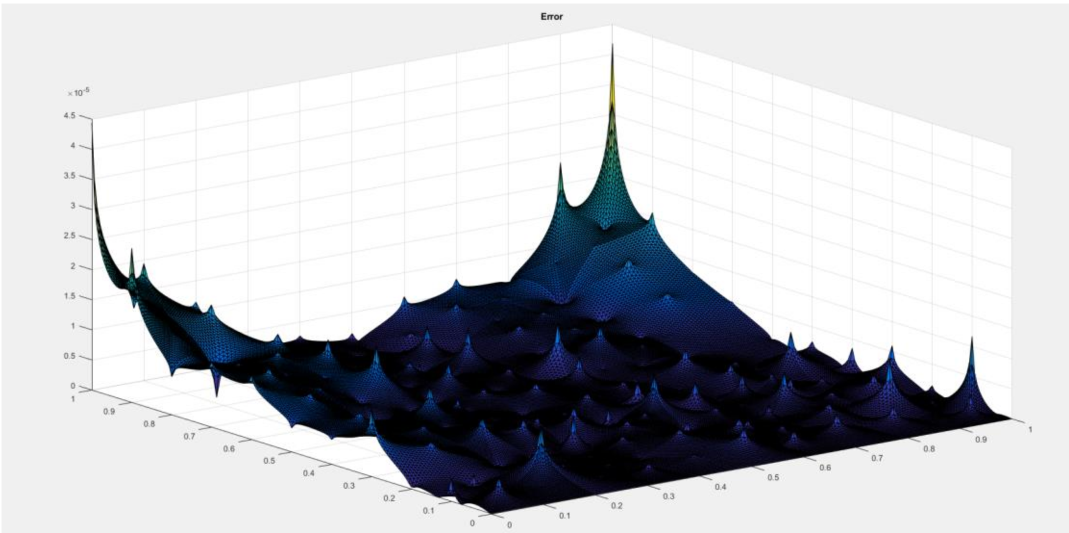

Figure 13. Absolute error between analytical and numerical solutions (t=50, mesh 5).

For part 3, so as to find a more effective way to obtain the solution at t=50, it is taken, the problem, under the assumption that the exponential term will be negligible when exp(-50) since this number is smaller and therefore, close to 0.

To do so, it has been considered the way mentioned before. This is by solving the parabolic partial differential equation. But, however, another method has been implemented.

The method relates the importance of dismissing and hence, deleting the exponential term and resolving the partial differential equation as if it was one of elliptic form. Since the time is yet not needed, this term goes off of the equation and it is solved by that way. Moreover, the dirichlet boundary condition that depends on the exponential term also gets reduced by neglecting that source term.

In the end, the results shown in the Chart 1 reflect that the idea of getting rid of the exponential term and solving the problem by means of approximating it to an elliptic pde works.