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# Computational Mechanics Tools

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## Assignment 2 MATLAB PDE Toolbox

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# 1 Transient Heat Equation

The following time dependent heat equation is solved with the MATLAB PDE toolbox:

$$u_t - \Delta u = f \quad \text{in } \Omega = [0, 1]^2,$$

where the source term is given by

$$f(x, y, t) = -3e^{-3t}.$$

Considering an initial condition at  $t = 0$  as

$$u(x, y, t = 0) = x^2 + xy - y^2 + 1,$$

and the boundary conditions as

$$\begin{aligned} u_n(x = 0, y, t) &= -y \\ u_n(x = 1, y, t) &= 2 + y \\ u_n(x, y = 0, t) &= x^2 + e^{-3t} \\ u_n(x, y = 1, t) &= x - 2 \end{aligned}$$

where  $u_n \equiv \partial u / \partial n$ .

The analytical solution of this problem is given by

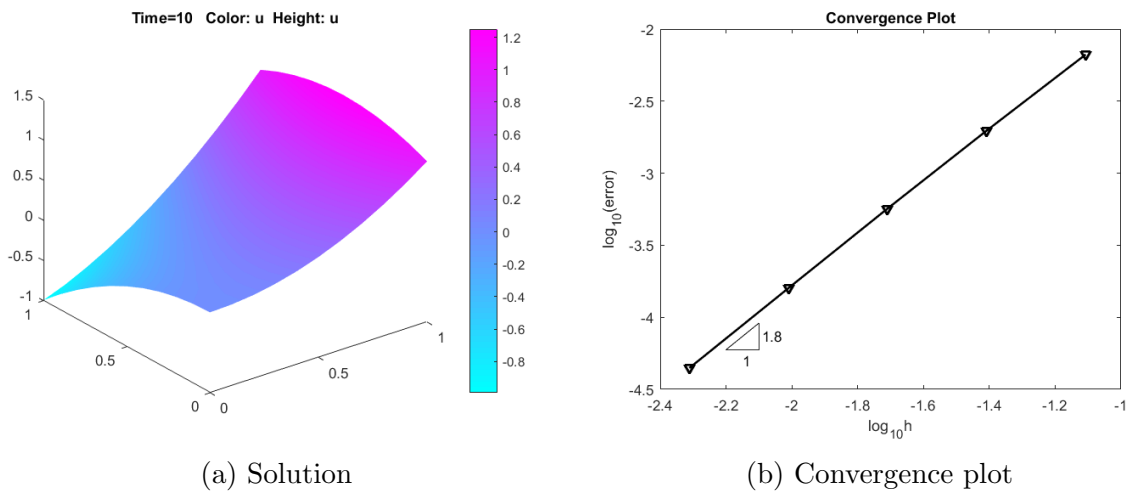
$$u(x, y, t) = x^2 + xy - y^2 + e^{-3t}.$$

## 1.1 (a) Solution at $t_{end} = 10$

Figure 1.1a shows the solution of the problem solved using PDE toolbox, considering the final time as  $t_{end} = 10$ . This is for a coarse mesh with 322 elements. This mesh is refined four times and at each refinement the  $\mathcal{L}_\infty$  error is computed as

$$e_{\mathcal{L}_\infty} = \max\{|u^h - u|\}$$

where,  $e_{\mathcal{L}_\infty}$  is the error in the  $\mathcal{L}_\infty$  norm,  $u^h$  is the numerically computed solution and  $u$  is the analytical solution. Figure 1.1b shows the convergence plot of this error, whereas the values are tabulated in Table 1.1. It can be seen that convergence rate is asymptotically approaching the expected rate of 2.

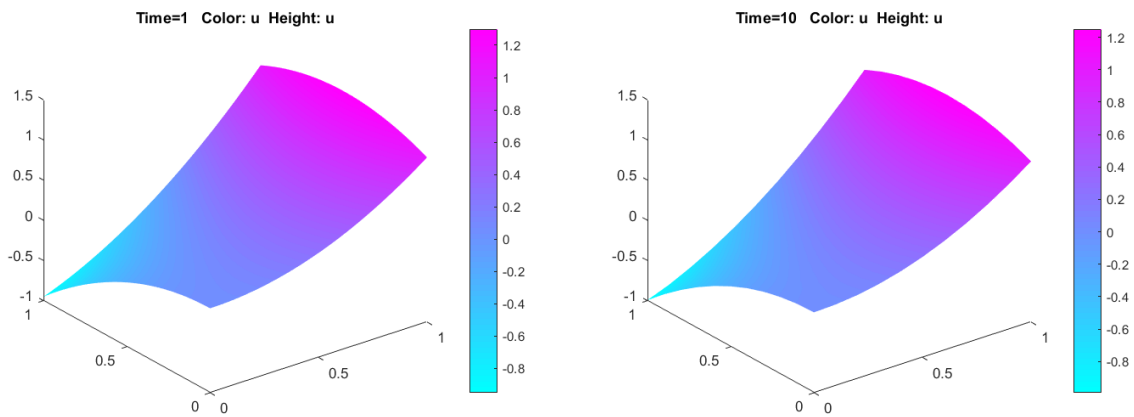
Figure 1.1: Plots at  $t_{end} = 10$ 

h	Error	Convergence Rate
0.0781	0.0067	1.75
0.0390	0.0020	1.80
0.0195	5.67e-04	1.83
0.0098	1.59e-04	1.84
0.0049	4.44e-05	

Table 1.1: Convergence table for error at  $t_{end} = 10$ 

## 1.2 (b) Comparison of solution at $t_{end} = 1$ and $t_{end} = 10$

Upon changing the final time, the solution varies slightly. In the plots of the solutions shown in Figure 1.2, it can be seen that a very small visible variation in the solution at  $(x, y) = (0, 1)$ . Also the maximum value of the solution is also different. In the case of time  $t_{end} = 1$ ,  $\max u^h = 1.2997$ , while for  $t_{end} = 10$ ,  $\max u^h = 1.2500$ . This is on par with the analytical solution which has an exponential term dependent on time. However, a more interesting feature is seen in the convergence plot shown in Figure 1.3. This data is obtained from solutions with 4 refinement of an initial mesh with 332 elements and a time step of  $\Delta t = 0.001$ . Here it can be seen that the solution for  $t_{end} = 1$  asymptotically approaches a convergence rate of 1 rather than the expected value of 2. Table 1.2 shows the values of the rates of convergence upon mesh refinement.



(a) Solution at  $t_{end} = 1$

(b) Solution at  $t_{end} = 10$

Figure 1.2: Comparison of solutions at  $t_{end} = 1$  and  $t_{end} = 10$

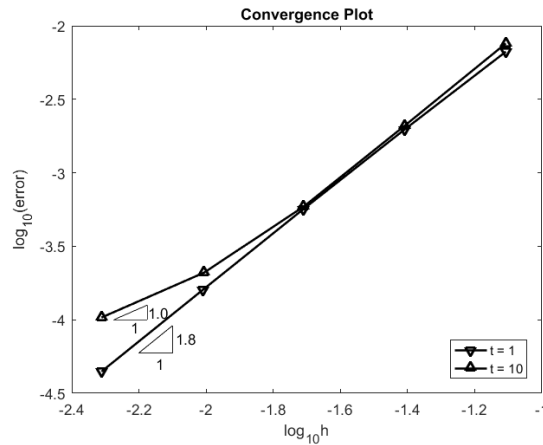


Figure 1.3: Comparison of convergence rates

h	Error	Convergence Rate
0.0781	0.0076	1.86
0.0390	0.0021	1.83
0.0195	5.88e-04	1.51
0.0098	2.09e-04	1.00
0.0049	1.04e-04	

Table 1.2: Convergence rates for error at  $t_{end} = 1$

### 1.3 (c) Solution at $t_{end} = 50$

In order to obtain the solution at time  $t_{end} = 50$  effectively, the problem type can be changed into an elliptic problem. This can be done as one can see from the analytical

solution that at  $t = 50$ , the solution effectively is independent of  $t$  as  $e^{-3*50} \approx 0$ . Figure 1.4a shows the numerically obtained solution considering the problem as an elliptic one and 1.4b shows the analytical solution. From 1.4 it can be seen that there is good agreement between the analytical and numerical solutions.

However, evidence that there is good agreement between the solutions can be assured from Figure 1.5a and Table 1.3 which show the convergence of the  $\mathcal{L}_\infty$  norm of error. It can be seen that  $e_{\mathcal{L}_\infty}$  convergences asymptotically at the expected rate of 2. Figures 1.4a, 1.4b and 1.5b have been plotted for a coarse mesh with 332 elements in the domain.

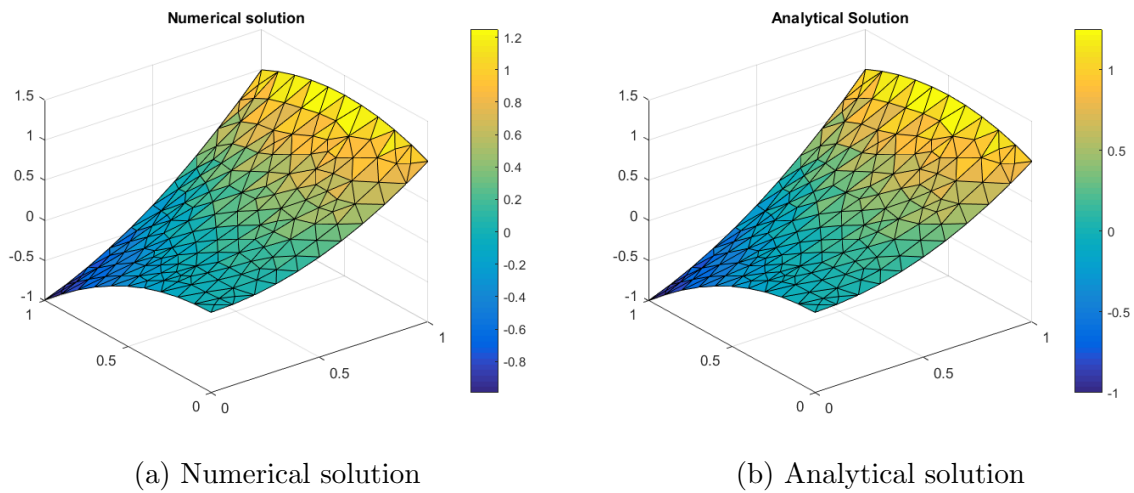


Figure 1.4: Solutions at  $t_{end} = 50$

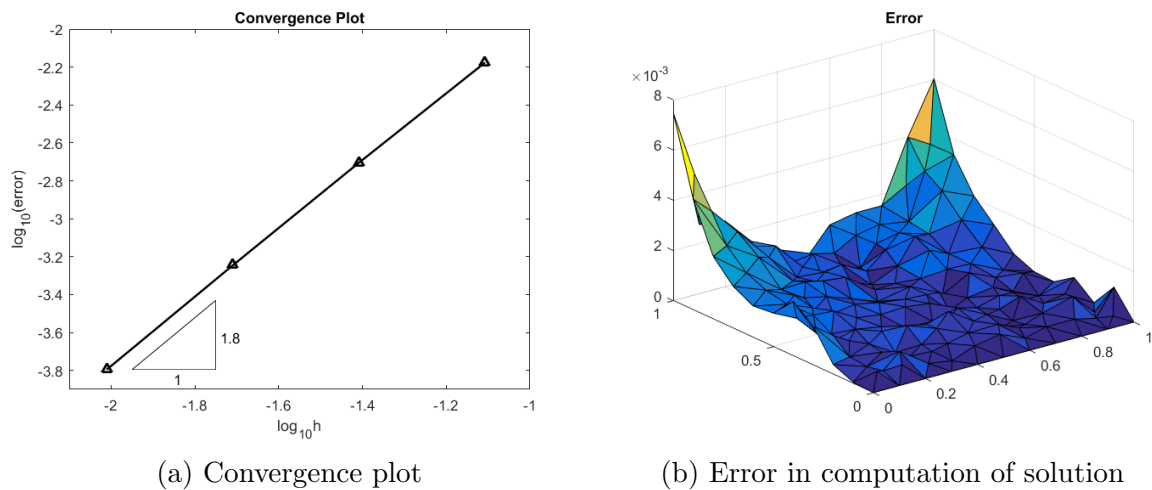


Figure 1.5: Convergence plot and error in solution at  $t_{end} = 50$

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h	Error	Convergence Rate
0.0781	0.0067	
0.0390	0.0020	1.75
0.0195	5.68e-04	1.80
0.0098	1.60e-05	1.83

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Table 1.3: Convergence rates for error at  $t_{end} = 50$