Universitat Politècnica de Catalunya

Computational Mechanical Tools

Master's Degree in Numerical Methods in Engineering

PDE-Toolbox on solving the heat equation (Third Assignment)

Author:
Pau Márquez

Supervisor:
Prof. A. Abdollahi

Academic Year 2019-2020

Contents

Abstract

The purpose of this text is to study the application of Matlab's PDE-Toolbox on the solution of the heat equation on a square domain with non-constant boundary and initial conditions. The solution will be found and conclusions will be extracted regarding the quality of the mesh, and its accuracy and convergence evaluated according to the mesh size. Eventually, the initial parabolic problem will be compared with the equilibrium solution when sufficient time is given, and the elliptic description of the problem applied instead when time-dependence does not play an important role.

1 Introduction

The problem to be solved is a parabolic heat equation with non-constant source term, initial condition and boundary conditions in a two-dimensional square domain. Specifically, the equation is

$$u_t - \vec{\nabla}^2 u = f \tag{1}$$

where f is the source term and depends on time in the form $f = e^{-3t}$.

(1) may be compared to the general PDE solved by Matlab:

$$d\frac{\partial u}{\partial t} + m\,\frac{\partial^2 u}{\partial t^2} - \vec{\nabla}\cdot(c\vec{\nabla}u) + au = f \tag{2}$$

Therefore, the coefficients to be specified to reach Eq. (1) are $d = 1, m = 0, c = 1$ and $a = 0$. Clearly this corresponds to a parabolic-type PDE. However, as will be seen further on, if time-dependence was to be removed (i,e, $d = 0$), the equation would become elliptic. The initial-value problem is defined at $t = 0$ through

$$u(x, y, t = 0) = x^2 + xy - y^2 + 1 \tag{3}$$

Eventually, the problem needs to be solved in a square domain with unity-length edges, as depicted in Fig. 2. Each of the edges of this domain has associated a boundary condition, which is dependent on the point coordinates and on time as shown in Table 1.

2 Results

The solution to the Parabolic PDE is obtained straightforward with Matlab's PDE Toolbox and its solution is shown in Fig. 2. It is interesting to note in Fig. 2a the boundary conditions being reinforced, meaning that the parabolas resulting from the Neuman conditions can be seen from the surface shape. The Dirichlet condition may be also observed, and its parabola is already giving indications that at $t = 10s$ the time gradient does not play an important

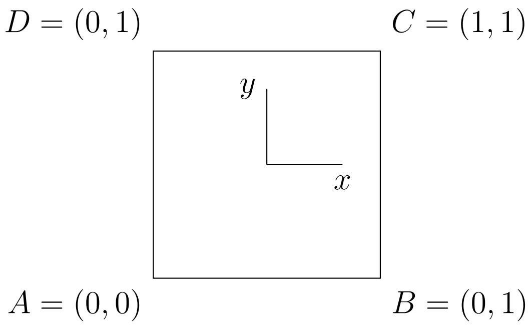


Figure 1: Schematic view of the domain in the coordinates x, y.

Edge	**Boundary condition**	**Equation**
AB	Dirichlet	$u(x, y, t) = x^2 + e^{-3t}$
BC	Newmann	$\frac{\partial u(x,y,t)}{\partial n} = 2 + y$
DC	Newmann	$\frac{\partial u(x,y,t)}{\partial n} = x - 2$
AD	Newmann	$\frac{\partial u(x,y,t)}{\partial n} = -y$

Table 1: Boundary conditions.

role in the solution. It is important to note that this surface plot has been generated with a 10465-element mesh, corresponding to a mesh element size of $0.01m$.

Given the analytical solution to the problem (also time-dependent), it is necessary to validate the solution by computing the maximum absolute error for all node values. The value obtained with this procedure is $1.66e - 4$, which is proof that the numerical approach is able to capture the behavior of the transient problem. If the mesh is refined and the logarithm of the maximum absolute error is plotted as a function of the element size h for four different refinements, a first-order polynomial is obtained showing that the accuracy improves with a reduction of the element size.

However, before examining the graph, which will contain the same polynomial for the elliptic case, let us study how is the solution affected as the final time increases. For this reason, Fig. 4 shows three maps containing the difference between nodal values for three different time increments, namely $10s - 15s$, $15s - 20s$ and $20s - 25s$. This plots will give an overview of the effect of time on the final solution.

When examining Fig.4, it is clear that, even for the first time increment, the solution is almost non-dependent on time, as the maximum difference is of the order of 10^{-13}, at least for times higher than $10s$. It is also interesting to note the absolute 0 error on the bottom edge, where the Dirichlet condition is imposed. The latter means that, if the solution to be found is at a

Mesh refinement level	Element size	Maximum absolute error	
		Parabolic PDE	**Elliptic PDE**
1	0.08	0.0075	0.0075
2	0.04	0.0021	0.0022
3	0.02	$5.97{\cdot}10^{-}4$	$6.19{\cdot}10^{-}4$
4	0.01	$1.66{\cdot}10^{-}4$	$1.73{\cdot}10^{-}4$
5	0.005	$4.80{\cdot}10^{-}5$	$4.80{\cdot}10^{-}5$
Slope of interpolating line		1.83	1.82

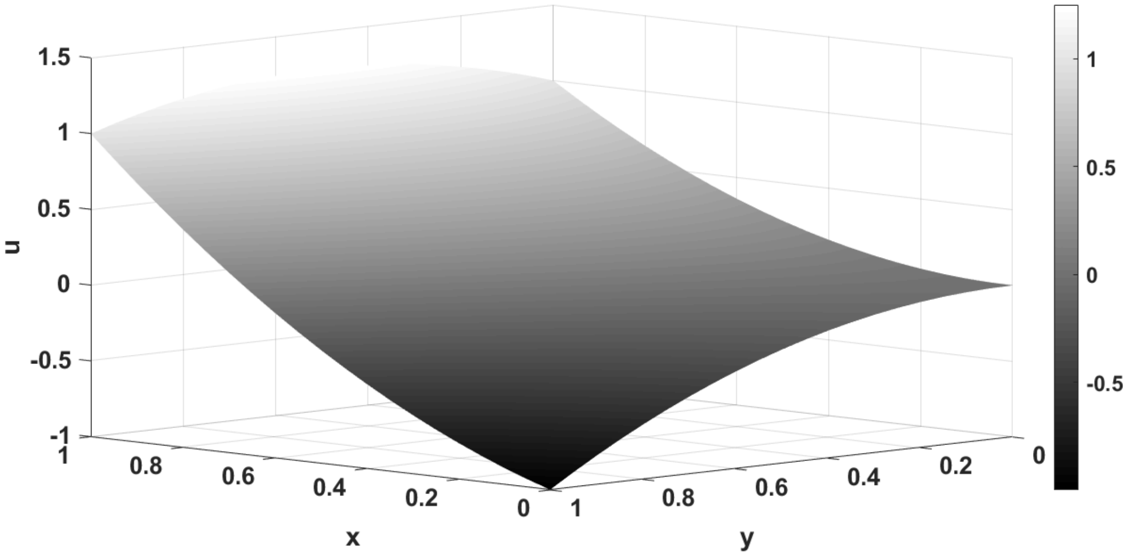
Table 2: Specific values for Fig. 3 and mean values of the slopes.

sufficient time value, the use of a parabolic PDE specification is no longer needed as the time dependence is non-existent, and an elliptic procedure may be used instead to solve the problem. This is the case at time $t_{end} = 50s$.

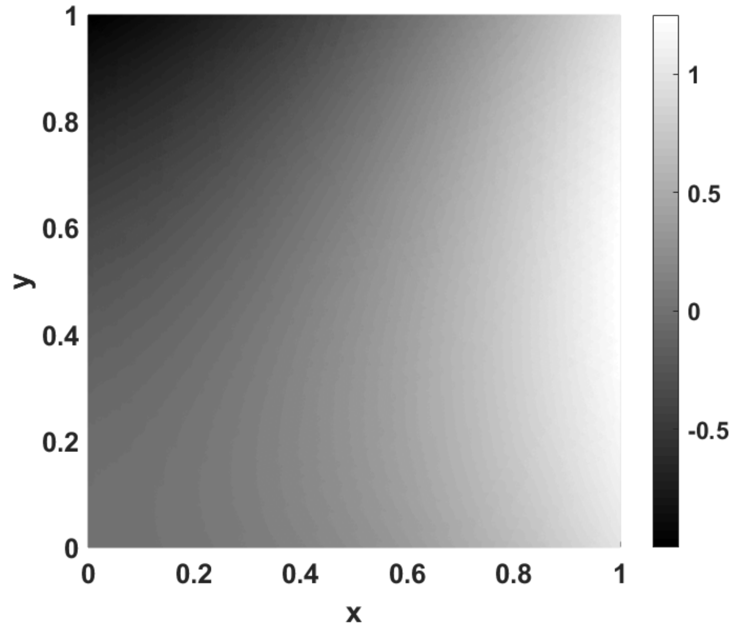
To provide numerical evidence of this new approach, it is included both a plot and a graphic which present the infinite norm for five different meshes using a parabolic and and elliptic PDE. As it can be seen, the evolution of the infinite norm is linear when plotted against the mesh element size in a logarithmic plot in both cases and the slopes of such lines are almost the same. Hence, the error behaves equally when improving the mesh and its values are almost the same, though the parabolic method performs better as expected.

3 Concluding remarks

The PDE-Toolbox has been successfully used to solve a heat transient problem with two different PDE specifications. The mesh accuracy has been improved by reducing the element size and the error order of convergence has been shown to be a line of slope close to 1.8. This way, the infinite norm decreases linearly in a logarithmic plot as a function of the element size. Moreover, it has been seen that a simpler and more effective method may be used instead if the time of the solution is high enough such that the time dependence may be neglected and therefore solve the problem as an elliptic case.



(a) Solution for equation 1 from perspective.



(b) Solution for equation 1 from above.

Figure 2: Solution for the PDE with a mesh size of $0.01m$.

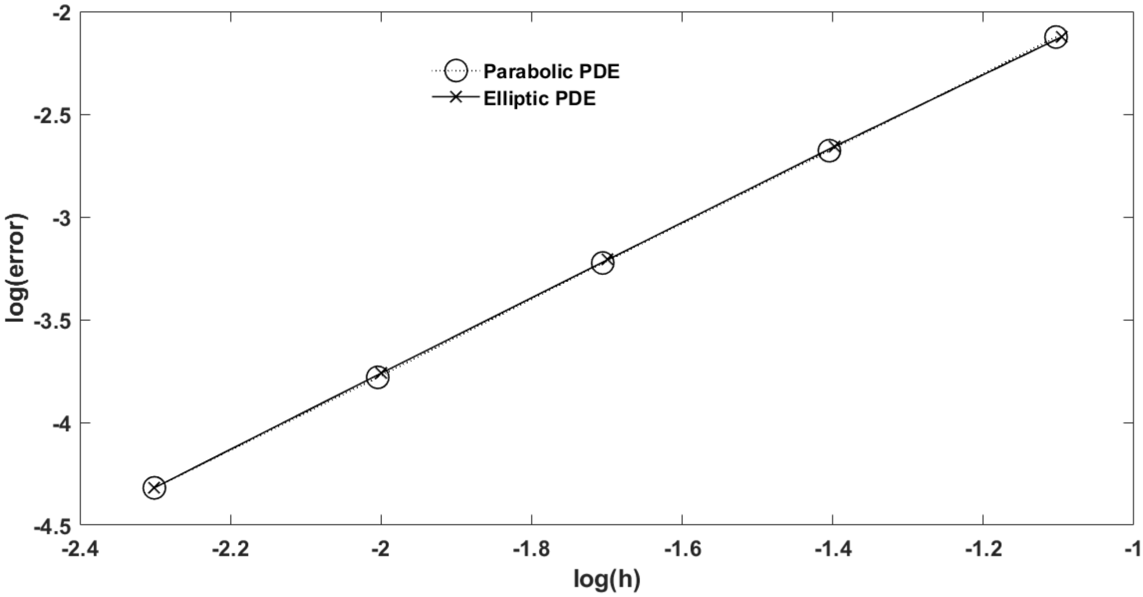
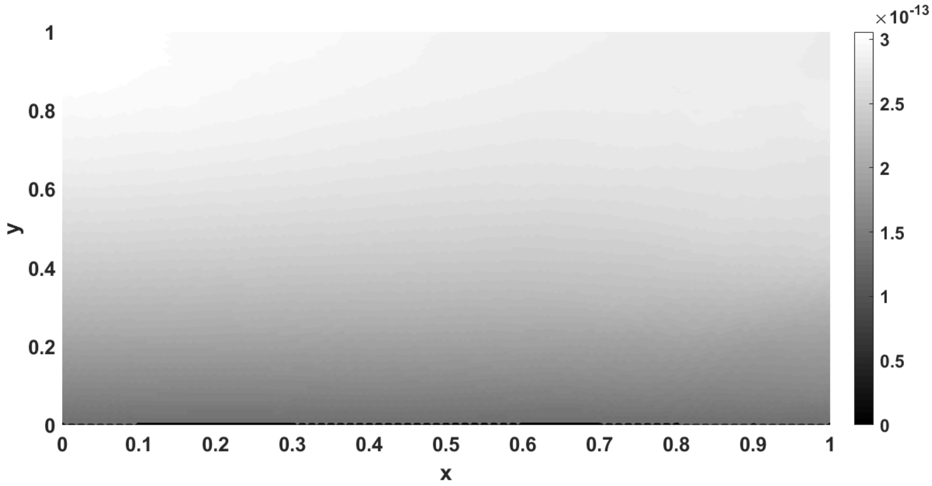
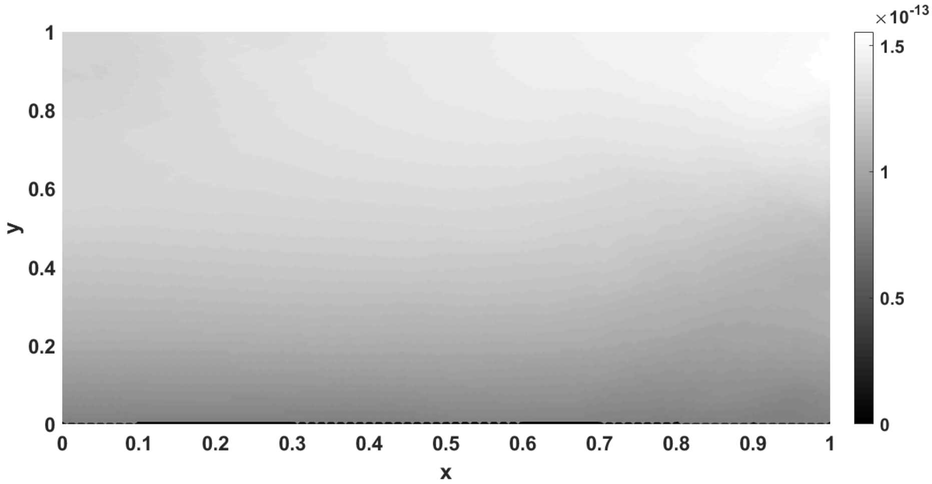


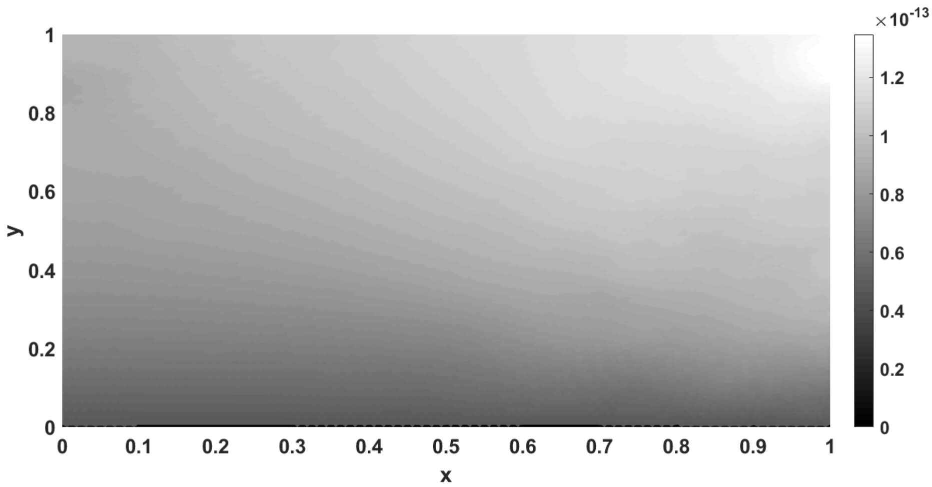
Figure 3: Logarithmic plot showing the linear evolution of the absolute maximum error for several element sizes.



(a) Absolute value of the difference between solutions at times $t = 15s$ and $t = 10s$.



(b) Absolute value of the difference between solutions at times $t = 20s$ and $t = 15s$.



(c) Absolute value of the difference between solutions at times $t = 25s$ and $t = 20s$.

Figure 4: Evolution of the parabolic PDE with time.