ADVANCED FLUID MECHANICS
Master of Science in Computational Mechanics/Numerical Methods
Fall Semester 2018

Homework 1: Dimensional Analysis, Governing Equations and Bernoulli Equation

Due date: November 14, 2018

Groups of two

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1. A solid sphere of radius a and density ρ_s is dropped into a container of liquid whose density and viscosity are ρ_l and μ_l. A short time after entering the liquid, the sphere us observed to descend into the liquid at a constant speed V_f. Derive a dimensionless expression for the dependence of V_f on the experimental variables a, ρ_s, ρ_l and μ_l and the gravity g.

2. The integral form of the second law of thermodynamics reads

$$\frac{D}{Dt}\int_{V_t} \rho s dV \geq -\int_{S_t} \frac{\boldsymbol{q}\cdot\boldsymbol{n}}{T} dS \tag{1}$$

where s is the entropy per unit mass.
The goal of this exercise is to show that the above inequality always holds under the following assumptions:

- Newtonian fluid with bulk viscosity ($K \geq 0$, $\mu > 0$)
- Fourier's law for heat conduction ($\boldsymbol{q} = -k\boldsymbol{\nabla}T$, $k > 0$)

(a) Apply Reynolds' transport theorem to inequality (1) and simplify the expression to obtain

$$\rho\frac{Ds}{Dt} + \nabla\cdot\frac{\boldsymbol{q}}{T} \geq 0.$$

(b) Use the relation $T\,ds = de + p\,d\left(\frac{1}{\rho}\right)$ to rewrite the term $\frac{Ds}{Dt}$ in terms of energy, pressure and density. Then, use energy and mass conservation to rewrite the inequality as

$$\boldsymbol{\sigma} : \nabla\boldsymbol{v} + p\nabla\cdot\boldsymbol{v} - \frac{\boldsymbol{q}\cdot\boldsymbol{\nabla}T}{T} \geq 0.$$

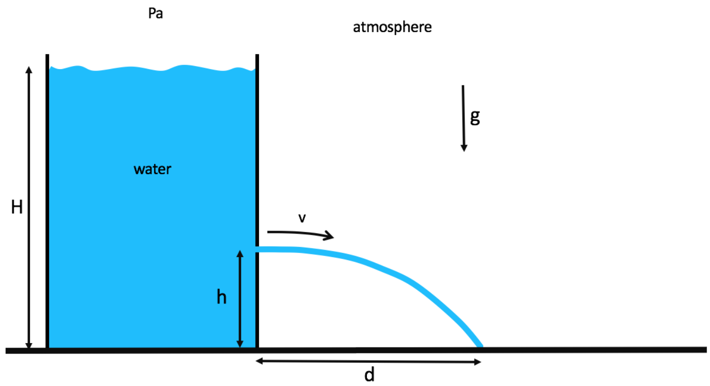
(c) Use the constitutive relation and Fourier's law to show that the inequality holds.

3. A deposit of height H, full of water and opened to the atmosphere, has an orifice at height h through which the water is being ejected steadily. The water streamline out of the deposit leaves the orifice horizontally and reaches the ground at a distance d from the deposit's basis.
Considering the water density ρ and a gravitational acceleration g,

(a) Obtain an expression for the velocity $\boldsymbol{v} = (u, w)$ of the ejected water stream.

(b) How does the distance d depend on the height h? Write the relation $d = d(h)$.

(c) For a value $H = 10m$, which is the orifice's height h that maximises the distance d?


Pa
atmosphere
g
water
H
v
h
d

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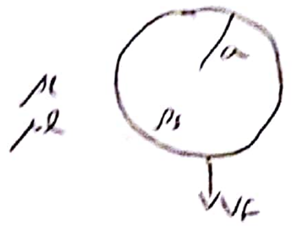

	V_f	a	ρ_s	ρ_l	μ_l	g
M	0	0	1	1	1	0
L	1	1	-3	-3	-1	1
T	-1	0	0	0	-1	-2

$\Rightarrow$ n: 3

$V_f = f(a, \rho_s, \rho_l, \mu_l, g)$

(V_f, a, ρ_s) $\begin{cases} \Pi_1 = V_f^a a^b \rho_s^c \rho_l \\ \Pi_2 = V_f^a a^b \rho_s^c \mu_l \\ \Pi_3 = V_f^a a^b \rho_s^c g \end{cases}$

$\Pi_1 = V_f^a a^b \rho_s^c \rho_l$

$(L^1T^{-1})^a (L^1)^b (M^1L^{-3})^c (M^1L^{-3}) = M^0L^0T^0$ $\begin{cases} c+1=0 \to c=-1 \\ a+b-3c-3=0 \to b=0 \\ -a=0 \to a=0 \end{cases}$ $\to \Pi_1 = \rho_s^{-1}\rho_l = \boxed{\dfrac{\rho_l}{\rho_s}}$

$\Pi_2 = V_f^a a^b \rho_s^c \mu_l$

$(L^1T^{-1})^a (L^1)^b (M^1L^{-3})^c (M^1L^{-1}T^{-1}) = M^0L^0T^0$ $\begin{cases} c+1=0 \to c=-1 \\ a+b-3c-1=0 \to b=-1 \\ -a-1=0 \to a=-1 \end{cases}$ $\to \Pi_2 = V_f^{-1} a^{-1} \rho_s^{-1} \mu_l$

$= \boxed{\dfrac{\mu_l}{\rho_s V_f a}} = \dfrac{1}{Re}$

Re

$\Pi_3 = V_f^a a^b \rho_s^c g$

$(L^1T^{-1})^a (L^1)^b (M^1L^{-3})^c (L^1T^{-2}) = M^0L^0T^0$ $\begin{cases} c=0 \\ a+b-3c+1=0 \to b=1 \\ -a-2=0 \to a=-2 \end{cases}$ $\to \Pi_3 = V_f^{-2} a g = \boxed{\dfrac{ag}{V_f^2}}$

Hypothesis:
if g = constant

Π_3: $\dfrac{a_m}{V_{fm}^2} = \dfrac{a_p}{V_{fp}^2} \to V_{fm} = V_{fp}\sqrt{\dfrac{a_m}{a_p}}$

Π_1: $\dfrac{\rho_{lm}}{\rho_{sm}} = \dfrac{\rho_{lp}}{\rho_{sp}}$

Π_2: $\dfrac{\mu_{lm}}{\rho_{sm} V_{fm} a_m} = \dfrac{\mu_{lp}}{\rho_{sp} V_{fp} a_p} \Rightarrow \dfrac{V_{fm}}{V_{fp}} = \sqrt{\dfrac{\mu_{lm}\,\rho_{sp}}{\mu_{lp}\,\rho_{sm}}}$

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$$\boxed{\frac{D}{Dt}\int_V \rho s\, dV \geq -\int_S \frac{q\cdot n}{T}\, dS}$$

- Newtonian fluid ($\kappa \geq 0$, $\mu > 0$)
- $q = -\kappa \nabla T$ ($\kappa > 0$)

a) Use transport theorem. (Deriving inside integral)

$$\int_V \frac{D(\rho s)}{Dt}\, dV + \int_S (\rho s)\, v\cdot n\, dS \geq -\int_S \frac{q\cdot n}{T}\, dS$$

+ Gauss $\boxed{\int_S \to \int_V}$ (twice)

Gauss theorem
$$\begin{cases} -\int_S \frac{q\cdot n}{T}\, dS = -\int_V \nabla\cdot\frac{q}{T}\, dV \\ \int_S (\rho s) v\cdot n\, dS = -\int_V \nabla\cdot(\rho s v)\, dV \end{cases}$$

$$\int_V \frac{D(\rho s)}{Dt}\, dV + \int_V \nabla\cdot(\rho s v)\, dV \geq -\int_V \nabla\cdot\left(\frac{q}{T}\right) dV$$

$$\frac{D(\rho s)}{Dt} \cancel{+} \nabla\cdot(\rho s v) + \nabla\cdot\frac{q}{T} \geq 0$$

↓ chain

$$s\frac{D\rho}{Dt} + \underbrace{\frac{Ds}{Dt}\rho + \rho s\, \nabla\cdot v}_{\rho\left[\frac{Ds}{Dt} + s\nabla\cdot v\right] = 0 \text{ conservation}} + \nabla\cdot\frac{q}{T} \geq 0$$

$$\boxed{s\frac{D\rho}{Dt} + \nabla\cdot\frac{q}{T} \geq 0}$$

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c)

$$\sigma : \nabla v + p \nabla \cdot v - \frac{q \cdot \nabla T}{T} \geq 0$$

Bulk viscosity

$$\begin{cases} K \geq 0 \\ \mu > 0 \\ k > 0 \end{cases} \quad K = \lambda + \frac{2}{3}\mu \rightarrow \lambda > 0$$

Use Fourier's law and constitutive relation (Newtonian)

$$\boxed{q = -k \nabla T}$$

$$\boxed{\sigma = -pI + \lambda \, tr(\nabla^s v) I + 2\mu \nabla^s v}$$

Using Fourier

$$\sigma : \nabla v + p \nabla \cdot v + \frac{k (\nabla T)^2}{T} \geq 0$$

$$(-pI + \lambda tr(\nabla^s v) I + 2\mu \nabla^s v) : \nabla v + p \nabla \cdot v + \frac{k (\nabla T)^2}{T} \geq 0$$

$$-pI : \nabla v + \lambda tr(\nabla^s v) I : \nabla v + 2\mu \underbrace{\nabla^s v : \nabla v}_{(\nabla \cdot v)^2} + p \nabla \cdot v + \frac{k(\nabla T)^2}{T} \geq 0$$

$$\cancel{-p \nabla \cdot v} + \lambda \underbrace{tr \nabla^s v}_{\nabla \cdot v} \nabla \cdot v + 2\mu (\nabla \cdot v)^2 + \cancel{p \nabla \cdot v} + \frac{k(\nabla T)^2}{T} \geq 0$$

$$\boxed{\lambda (\nabla \cdot v)^2 + 2\mu(\nabla \cdot v)^2 - \frac{k(\nabla T)^2}{T} \geq 0}$$

$\lambda \geq 0$, $(\nabla \cdot v)^2 \geq 0$, $\mu \geq 0$, $(\nabla \cdot v)^2 \geq 0$, $k \geq 0$, $(\nabla T)^2 \geq 0$, $T > 0$

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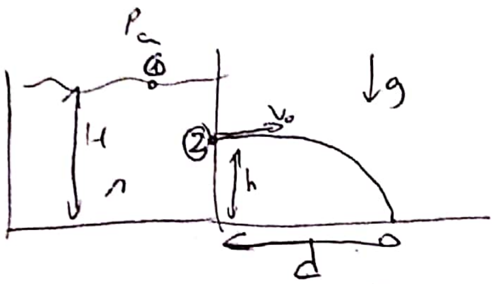
3) =



a) $v=(u,w)$

$u=v_0$

$\frac{\partial w}{\partial t}=g \rightarrow w=gt+C = gt$

$t=0, w=0$

$C=0$

$\boxed{V=(V_0, gt)}$

b) $d(h)$?

• Bernoulli, steady 1-2 : $P_2=P_a$ $V_1 \simeq 0$

$$\int_1^2 \frac{\partial V}{\partial t}\cdot ds + \frac{V_0^2}{2} + \cancel{\frac{P_2}{\rho}} + z_2 = \cancel{\frac{P_a}{\rho}} + z_1 + \cancel{\frac{V_1^2}{2}}$$

$$\int_1^2 \frac{\partial V}{\partial t}\cdot ds + \frac{V_0^2}{2} - g(H-h)=0$$

$\simeq 0$ (wide)

$$\frac{V_0^2}{2}=g(H-h)$$

$$V_0=\sqrt{2g(H-h)}$$

$$\begin{cases} x=\int v_x dt = \int V_0 dt = V_0 t + C = \boxed{V_0 t} = \boxed{\sqrt{2g(H-h)}\, t} \quad (t=0, x=0;\ C=0) \\ y=\int v_y dt = g\frac{t^2}{2}+C = \boxed{g\frac{t^2}{2}+h} \quad (t=0,\ y=h) \end{cases}$$

$$\boxed{d(h)=x_f(h)}$$

when $y=0$ $\rightarrow t_f=\sqrt{\frac{2h}{g}} \rightarrow x_f=\sqrt{2g(H-h)}\cdot t_f = \sqrt{2\cancel{g}(H-h)\frac{2h}{\cancel{g}}} = 2\sqrt{(H-h)h}$

$$\boxed{d(h)=2\sqrt{(H-h)h}}$$

c) for $H=10\,m$

¿h that maximizes d?

$$\frac{\partial d(h)}{\partial h}=0 \rightarrow \frac{\partial}{\partial h}\left(\sqrt{(H-h)h}\right)=0 \rightarrow \frac{H-2h}{2\sqrt{(H-h)h}}=0 \rightarrow H=2h$$

$$\boxed{h_{max}=\frac{H}{2}}$$

for $H=10$: $h=5m$

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