ADVANCED FLUID MECHANICS
Homework 1 : Mathematical Preliminaries and Governing Equations

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1. Proof of Vector Identities

   (a) $\nabla \cdot (\nabla \times \vec{F}) = 0$
   Solution :

$$\begin{aligned}\nabla \cdot (\nabla \times \vec{F}) &= (\nabla \times \vec{F})_{i,i} \\ &= (\epsilon_{ijk} F_{k,j})_{,i} \\ &= \epsilon_{ijk} F_{k,ji} \\ &= 0\end{aligned}$$

   (b) $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$
   Solution :

$$\begin{aligned}[\nabla \times (\nabla \times \vec{F})]_i &= \epsilon_{ijk} (\nabla \times \vec{F})_{k,j} \\ &= \epsilon_{ijk} (\epsilon_{kpq} F_{q,p})_j \\ &= \epsilon_{ijk} \epsilon_{kpq} F_{q,pj} \\ &= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) F_{q,pj} \\ &= F_{j,ij} - F_{i,jj} \\ &= (F_{j,j})_{,i} - (F_i)_{,jj} \\ &= [\nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}]_i\end{aligned}$$

   Hence, proved.

   (c) $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$
   Solution :

$$\begin{aligned}\nabla \cdot (\vec{F} \times \vec{G}) &= (\vec{F} \times \vec{G})_{i,i} \\ &= (\epsilon_{ijk} F_j G_k)_{,i} \\ &= \epsilon_{ijk} F_{j,i} G_k + \epsilon_{ijk} G_{k,i} F_j \\ &= \epsilon_{ijk} F_{j,i} G_k - \epsilon_{ikj} G_{k,i} F_j \\ &= (\nabla \times \vec{F})_k (\vec{G})_k - (\nabla \times \vec{G})_j (\vec{F})_j \\ &= (\nabla \times \vec{F}) \cdot \vec{G} - (\nabla \times \vec{G}) \cdot \vec{F}\end{aligned}$$

2. 2^{nd} Law of Thermodynamics for Newtonian Fluids

   The Gibbs equation, obtained by combining the first and second laws of thermodynamics, is :

$$T ds = de + p\, dv$$

   Gibbs equation can be written for a moving fluid as :

$$T\frac{Ds}{Dt} = \frac{De}{Dt} + p\frac{Dv}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

Now, the rate of change of energy, De/Dt, can be expressed in terms of viscous dissipation and heat flux using the following equation.

$$\rho\frac{De}{Dt} = -p\nabla\cdot\vec{v} - \nabla\cdot\vec{q} + \Phi$$

where, $\Phi = \lambda(\nabla\cdot\vec{v})^2 + 2\mu\nabla^S\vec{v} : \nabla\vec{v}$

Eliminating De/Dt using the above two equations, we have

$$T\frac{Ds}{Dt} = \frac{1}{\rho}(-p\nabla\cdot\vec{v} - \nabla\cdot\vec{q} + \Phi) - \frac{p}{\rho^2}\frac{D\rho}{Dt}$$
$$\implies \rho T\frac{Ds}{Dt} = -\frac{p}{\rho}(\nabla\cdot\vec{v} + \frac{D\rho}{Dt}) - \nabla\cdot\vec{q} + \Phi$$

Using the mass conservation equation, the expression within paranthesis is 0. Hence,the equation reduces to,

$$\implies \rho\frac{Ds}{Dt} = -\frac{\nabla\cdot\vec{q}}{T} + \frac{\Phi}{T}$$

Using the vector identity, $\nabla\cdot(\frac{\vec{q}}{T}) = \frac{\nabla\cdot\vec{q}}{T} - \frac{\vec{q}}{T^2}\cdot\nabla T$

$$\implies \rho\frac{Ds}{Dt} = -\nabla\cdot\left(\frac{\vec{q}}{T}\right) - \frac{\vec{q}}{T^2}\cdot\nabla T + \frac{\Phi}{T}$$

Using the constitutive equation for heat flux, $\vec{q} = -k\nabla T$, we get

$$\implies \rho\frac{Ds}{Dt} = -\nabla\cdot\left(\frac{\vec{q}}{T}\right) + k\nabla T\cdot\nabla T\left(\frac{1}{T}\right)^2 + \frac{\Phi}{T}$$
$$\implies \rho\frac{Ds}{Dt} = -\nabla\cdot\left(\frac{\vec{q}}{T}\right) + k\left(\frac{||\nabla T||}{T}\right)^2 + \frac{\Phi}{T}$$

For a positive k, the second term of the RHS is positive, which is always (otherwise heat would flow up the temperature gradients and that would be a violation of the 2nd law of thermodynamics). The third term of RHS can be proven to be positive as follows:

$$
\begin{aligned}
\Phi &= \lambda(\nabla\cdot\vec{v})^2 + 2\mu\nabla^S\vec{v} : \nabla\vec{v} \\
&= \lambda(\nabla\cdot\vec{v})^2 + 2\mu\nabla^S\vec{v} : \left[\frac{\nabla^S\vec{v} + (\nabla^A\vec{v})}{2}\right] \\
&= \lambda(\nabla\cdot\vec{v})^2 + 2\mu\nabla^S\vec{v} : \nabla^S\vec{v} \\
&= \left(K - \frac{2}{3}\mu\right)(\nabla\cdot\vec{v})^2 + 2\mu\nabla^S\vec{v} : \nabla^S\vec{v} \\
&= \left(K - \frac{2}{3}\mu\right)\left(\sum_i \frac{dv_i}{dx_i}\right)^2 + 2\mu\left[\sum_i\left(\frac{dv_i}{dx_i}\right)^2 + \sum_{i,j}^{i\neq j}\left(\frac{dv_i}{dx_j} + \frac{dv_j}{dx_i}\right)^2\right]
\end{aligned}
$$

rearranging the terms

$$
= \underbrace{K\left(\sum_i \frac{dv_i}{dx_i}\right)^2 + 2\mu\left[\sum_{i,j}^{i\neq j}\left(\frac{dv_i}{dx_j} + \frac{dv_j}{dx_i}\right)^2\right]}_{A} + \underbrace{2\mu\left[\sum_i\left(\frac{dv_i}{dx_i}\right)^2\right] - \frac{2}{3}\mu\left(\sum_i \frac{dv_i}{dx_i}\right)^2}_{B}
$$

Consider the parts A and B of the above expression seperately. A is always positive for $K > 0$ and $\mu > 0$. Reducing B further:

$$
\begin{aligned}
B &= 2\mu\left[\sum_i\left(\frac{dv_i}{dx_i}\right)^2\right] - \frac{2}{3}\mu\left(\sum_i \frac{dv_i}{dx_i}\right)^2 \\
\implies \frac{B}{2\mu} &= \sum_i\left(\frac{dv_i}{dx_i}\right)^2 - \frac{1}{3}\left(\sum_i \frac{dv_i}{dx_i}\right)^2 \\
&= \frac{2}{3}\sum_i\left(\frac{dv_i}{dx_i}\right)^2 - \frac{1}{3}\left(\sum_{i,j}^{i\neq j}\frac{dv_i}{dx_i}\frac{dv_j}{dx_j}\right) \\
&= \frac{2}{3}\sum_i\left(\frac{dv_i}{dx_i}\right)^2 - \frac{1}{3}\left(\sum_{i,j}^{i\neq j}\frac{dv_i}{dx_i}\frac{dv_j}{dx_j}\right)
\end{aligned}
$$

Using $2(a^2 + b^2 + c^2 - ab - bc - ca)$
$= (a - b)^2 + (b - c)^2 + (c - a)^2$

$$
= \frac{1}{3}\sum_{i,j}^{i\neq j}\left(\frac{dv_i}{dx_i} - \frac{dv_j}{dx_j}\right)^2 \geq 0
$$

Since, both expressions A and B are proven to be positive, it is safe to say $\Phi \geq 0$

Hence, the thermodynamic equation reduces to the following inequation :

$$
\begin{aligned}
&\implies \rho\frac{Ds}{Dt} \geq -\nabla\cdot\left(\frac{\vec{q}}{T}\right) \\
&\implies \int_{V_t}\rho\frac{Ds}{Dt}dV \geq -\int_{V_t}\nabla\cdot\left(\frac{\vec{q}}{T}\right)dV
\end{aligned}
$$

Using the Gausss divergence theorem on the RHS of the inequation, we get

$$\implies \int_{V_t} \rho \frac{Ds}{Dt} dV \geq -\int_{S_t} \frac{\vec{q} \cdot \hat{n}}{T} dS$$

Using Reynold's Lemma,

$$\implies \frac{D}{Dt} \int_{V_t} \rho s dV \geq -\int_{S_t} \frac{\vec{q} \cdot \hat{n}}{T} dS$$

Hence, proved.