Advanced Fluid Mechanics

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ANIL BETTADAHALLI CHANNAKESHAVA

HOME WORK : 2

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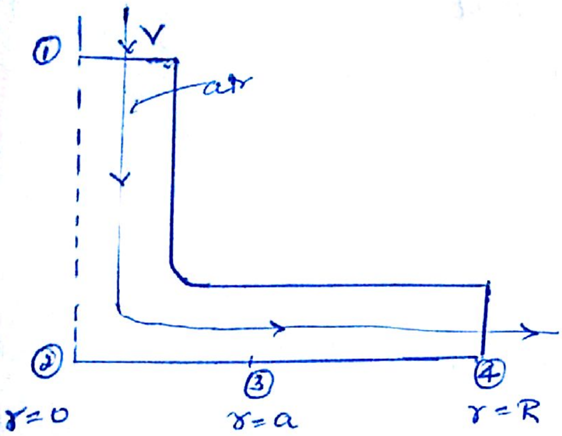
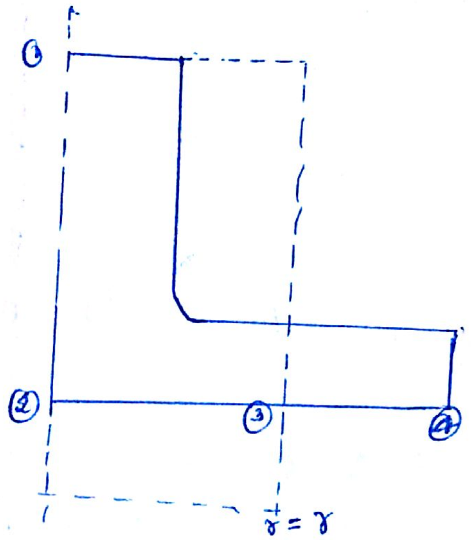
(1)



Taking the following control volume



Applying mass conservation in Reynolds transport theorem form,

$$\frac{DM_{sys}}{Dt} = 0 = \frac{d}{dt}\int_{cv} \rho \, dV + \int_{cs} \rho \bar{V}\cdot\bar{n} \, ds$$

Assuming steady state

$$0 = \int_{cs} \rho \bar{V}\cdot\bar{n} \, ds$$

Assuming constant density & using $V_r = f(r)$

$$0 = -\rho V \pi a^2 + 2\rho \pi r h V_r$$

$$\Rightarrow \boxed{V_r = \left(\frac{a^2 V}{2h}\right)\frac{1}{r}}$$

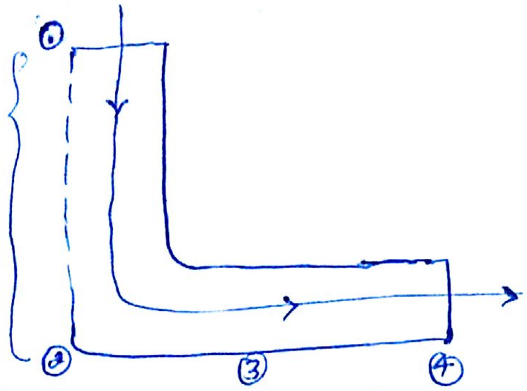
so for $r \geq a$, $V_r = \frac{C_2}{r}$, where $\boxed{C_2 = \frac{a^2 V}{2h}}$ (1)

The velocity field should be continous at $r = a$ since there is no mass getting added / removed at that location.

$$\Rightarrow V_r \text{ for } r \le a \Big|_{r=a} = V_r \text{ for } r \ge a \Big|_{r=a}$$

$$-C_1 \cancel{a} = \frac{a^2 V}{2h\cancel{a}}$$

$$\Rightarrow \boxed{C_1 = -\frac{V}{2h}}$$



* Applying Bernoulli theorem b/w ① & ②. since flow is inviscid & assumed to incompressible & with constant 'ρ'

$$\rho \int_1^2 \underbrace{\frac{\partial \bar{V}}{\partial t}}_{\to 0 \text{ steady state}} \cdot d\bar{l} + P_2 + \frac{1}{2}\rho \underbrace{V_2^2}_{V_2 = 0 \text{ at stagnation point}} - P_1 - \frac{1}{2}\rho V_1^2 - \underbrace{\rho g h}_{\rho_{air} \text{ is less \& neglecting term}} = 0$$

$$\Rightarrow \boxed{P_2 = P_1 + \frac{1}{2}\rho V^2}$$

P_1 is given to be P_0

* Applying Bernoulli's theorem b/w ② & ③ with same assumptions as above. $r < a$

$$P_2 + \frac{1}{2}\rho \underbrace{V_2^2}_{\text{stagnation } V_2 = 0} + 0 = P_r + \frac{1}{2}\rho V_r^2 + \rho g \underbrace{H}_{\text{small height}}$$

$$P_1 + \frac{1}{2}\rho V^2 = P_r + \frac{1}{2}\rho\left(\frac{Vr}{2h}\right)^2$$

$$P_r - P_1 = \frac{1}{2}\rho V^2\left(1 - \frac{r^2}{4h^2}\right)$$

$$\boxed{P_{r,\text{gauge}} = \frac{1}{2}\rho V^2\left(1 - \frac{r^2}{4h^2}\right)} \quad r < a$$

Applying Bernoulli theorem b/w 2 & 4 with same assumptions as above. $(r \geq a)$

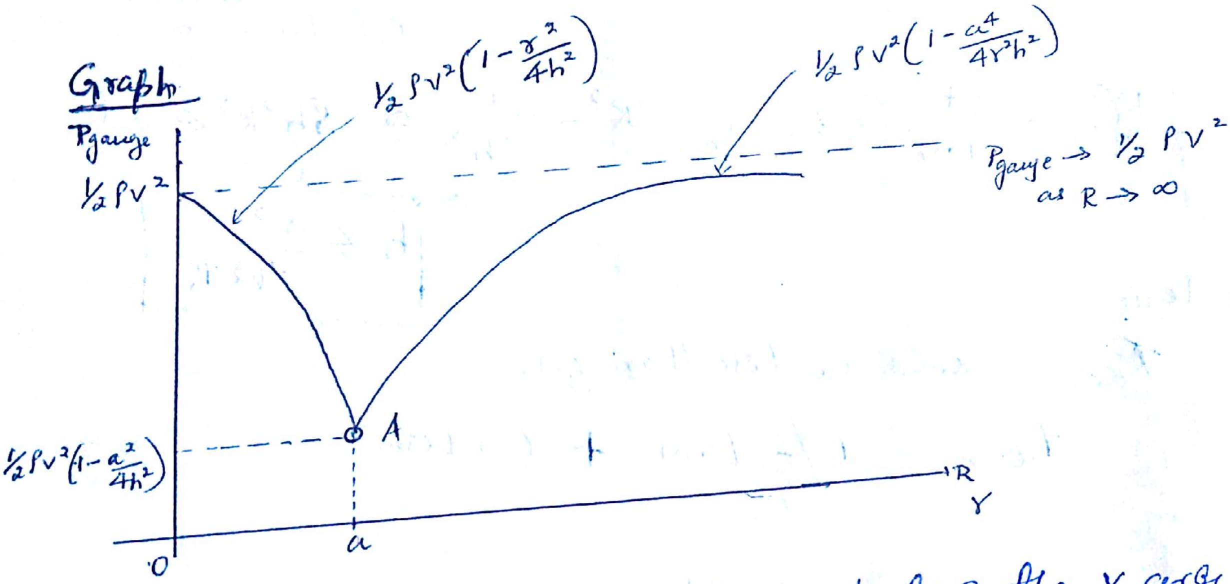
$$P_2 = P_r + \frac{1}{2}\rho V_r^2$$

$$P_1 + \frac{1}{2}\rho V^2 = P_r + \frac{1}{2}\rho\left(\frac{a^2 V}{2rh}\right)^2$$

$$P_r - P_1 = \frac{1}{2}\rho V^2\left[1 - \frac{a^4}{4h^2 r^2}\right]$$

$$\boxed{P_{r,\,gauge} = \frac{1}{2}\rho V^2\left(1 - \frac{a^4}{4r^2h^2}\right) \quad r \geq a}$$

Graph



The point A could be above or below the x axis depending on the value of $1 - \frac{a^2}{4h^2}$

if $1 - \frac{a^2}{4h^2} < 0$, it goes below x axis,

But the overall nature & behaviour shape remains same.

Calculating force on cardboard disk.

Total downward force acting on disk

$$= \int_0^R P_r \, 2\pi r \, dr$$

$$= \int_0^a \frac{1}{2}\rho V^2\left[1 - \frac{r^2}{4h^2}\right] 2\pi r\, dr + \int_a^R \frac{1}{2}\rho V^2\left(1 - \frac{a^4}{4r^2h^2}\right) 2\pi r\, dr$$

$$= \frac{2\pi}{2} \rho V^2 \left[\cancel{\frac{a^2}{2}} - \frac{a^4}{16h^2} \right] + \frac{1}{2} \rho V^2 (2\pi) \left[\left(\frac{R^2}{2} - \cancel{\frac{a^2}{2}} \right) - \frac{a^4}{4h^2} \ln\left(\frac{R}{a}\right) \right]$$

$$\boxed{\downarrow F_{total} = \pi \rho V^2 \left[\frac{R^2}{2} - \frac{a^4}{16h^2} \right] + \pi \rho V^2 \left(\frac{a^4}{4h^2} \right) \ln \frac{a}{R}}$$

$$F_{down} = \pi \rho V^2 \left[\frac{R^2}{2} - \frac{a^4}{16h^2} \right. + \pi \rho V^2 \left(\frac{a^4}{4h^2} \right) \ln \left(\frac{a}{R} \right)$$

(-ve / +ve)

always (-ve) since $a < R$ & Rest terms are positive

$$\frac{R^2}{2} - \frac{a^4}{16h^2} < 0 \text{, if } R^2 < \frac{a^4}{8h^2} \text{ or } 8h^2R^2 < a^4$$

or $\boxed{h < \frac{a^2}{2\sqrt{2} R}}$

Hence,

F_{down} could be less than zero

$$F_{down} = \underset{+ve/-ve}{(+/-)\text{ term}} + \underset{(-ve)}{(-)\text{ term}}$$

In the given problem

$a = 1E-2$ m

$R = 5E-2$ m

$h = 0.1E-2$ m

$M = 10E-3$ kg

F_{down} + weight$_{disk}$ = 0

F_{up} = weight$_{disk}$

$$10E-3 \times 9.81 = \frac{\pi}{2} \rho V^2 \left[(5E-2)^2 - \frac{(1E-2)^4}{8(0.1E-2)^2} + \frac{(1E-2)^4}{2(0.1E-2)^2} \ln \frac{(1E-2)}{(5E-2)} \right]$$

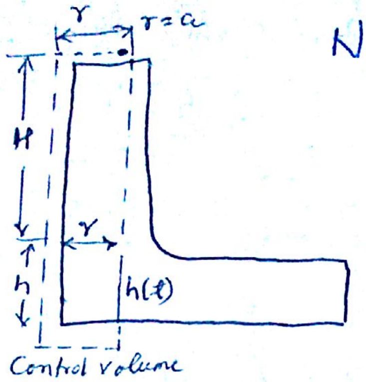
$\rho_{air} = 1.225$ kg/m3

$$10E-3 \times 9.81 = \frac{\pi}{2} \left[1.225 V^2 \right] \times 6.7972E-3$$

$$10E-3 \times 9.81 = 0.013079 \, V^2$$

$$\boxed{V = 2.7386 \text{ m/s}}$$

Scanned by CamScanner



No longer steady state problem.

* For $r \leq a$, Mass Continuity in Reynolds Transport Theorem

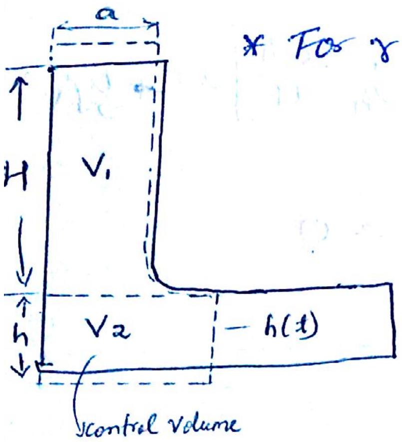
$$\frac{DM_{sys}}{Dt} = \frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho \bar{v} \cdot \bar{n} \, ds$$

$$0 = \frac{d}{dt}\left(\rho \pi r^2 (h+H)\right) + \rho V_r 2\pi r h - \rho V \pi r^2$$

assuming $\rho =$ constant & H doesn't change

$$0 = \pi r^2 \frac{dh}{dt} + V_r h 2\pi r - V\pi r^2$$

$$\boxed{V_r = \frac{Vr}{2h} - \frac{r}{2h}\frac{dh}{dt} \qquad r \leq a}$$



* For $r \geq a$, mass continuity in Reynolds Transport theorem. & $\rho =$ constant

$$\frac{DM_{sys}}{Dt} = \frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho \bar{v} \cdot \bar{n} \, ds$$

$$0 = \frac{d}{dt}\left(V_1 + V_2\right) + 2\pi r h V_r - V\pi a^2$$

$$\frac{dV_1}{dt} = 0 \ , \quad \frac{dV_2}{dt} = \pi r^2 \frac{dh}{dt}$$

$$\pi r^2 \frac{dh}{dt} = V\pi a^2 - 2\pi r h V_r$$

$$\boxed{V_r = \frac{Va^2}{2rh} - \frac{r}{2h}\frac{dh}{dt} \qquad r \geq a}$$

using the expressions and applying Bernoulli b/w
$r = a$ to $r = R$ (2 to 4)

$$\int_0^R \frac{d\bar{v}}{dt}\cdot d\bar{l} + P_R + \frac{1}{2}\rho V_R^2 - P_2 - \frac{1}{2}\rho \underbrace{V_2^2}_{\to 0 \text{ (stagnation) } V_2 = 0} = 0$$

given $P_R = P_0$ & $P_2 = P_1 + \frac{1}{2}\rho V^2$, $P_1 = P_0$

$$\therefore \quad P_2 = P_0 + \frac{1}{2}\rho V^2$$

$$P_R - P_2 = -\frac{1}{2}\rho V^2$$

$$V_R = \frac{Va^2}{2Rh} - \frac{R}{2h}\left(\frac{dh}{dt}\right)$$

$$\boxed{\int_0^a \frac{d\bar{v}}{dt}\cdot d\bar{l} + \int_a^R \frac{d\bar{v}}{dt}\cdot d\bar{l} - \frac{1}{2}\rho V^2 + \frac{1}{2}\rho\left[\frac{Va^2}{2Rh} - \frac{R}{2h}\frac{dh}{dt}\right]^2}$$

$$\therefore \Rightarrow \int_0^a \rho\left[\frac{-Vr}{2h^2}\frac{dh}{dt} + \frac{r}{2h^2}\left(\frac{dh}{dt}\right)^2 - \frac{r}{2h}\frac{d^2h}{dt^2}\right]dr +$$

$$\int_0^R \rho\left[-\frac{Va^2}{2rh^2}\frac{dh}{dt} + \frac{r}{2h^2}\left(\frac{dh}{dt}\right)^2 - \frac{r}{2h}\frac{d^2h}{dt^2}\right]dr - \frac{1}{2}\rho V^2$$

$$+ \frac{1}{2}\rho\left[\frac{Va^2}{2Rh} - \frac{R}{2h}\frac{dh}{dt}\right]^2 = 0$$

(HW-2)

② Consider a 2D uniform flow ($v = Ue_x$) past a cylinder of radii R located at the origin of coordinates. Assume that the fluid is incompressible and inviscid so that velocity can be defined using a stream function & Bernoulli's equation applies.

Solution:

Laplacian in polar co-ordinates.

$$\nabla^2 a = \frac{\partial^2 a}{\partial r^2} + \frac{1}{r}\frac{\partial a}{\partial r} + \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2}$$

Given:

$$\psi(r,\theta) = f(r)\sin\theta \quad \text{with } f(r) = r^{\alpha}$$

$$\psi = r^{\alpha}\sin\theta$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} \quad \text{— (i)}$$

$$\frac{\partial\psi}{\partial r} = \alpha r^{\alpha-1}\sin\theta, \quad \frac{\partial^2\psi}{\partial r^2} = (\alpha^2-\alpha) r^{\alpha-2}\sin\theta$$

$$\frac{\partial\psi}{\partial\theta} = r^{\alpha}\cos\theta, \quad \frac{\partial^2\psi}{\partial\theta^2} = -r^{\alpha}\sin\theta$$

Sub in (i).

$$\nabla^2\psi = (\alpha^2-\alpha) r^{\alpha-2}\sin\theta + \frac{1}{r}\alpha r^{\alpha-1}\sin\theta + \frac{1}{r^2}\,{-r^{\alpha}}\sin\theta$$

$$= (\alpha^2-\alpha) r^{\alpha-2}\sin\theta + \alpha r^{\alpha-2}\sin\theta + r^{\alpha-2}\sin\theta$$

$$\nabla^2\psi = r^{\alpha-2}\sin\theta\left[\alpha^2 - \cancel{\alpha} + \cancel{\alpha} + 1\right]$$

$$\Delta\psi = \nabla^2\psi = r^{\alpha-2}\sin\theta\left[\alpha^2-1\right]$$

As we require plane flow to be irrotational

$$\Delta\psi = 0$$

$$r^{\alpha-2}\sin\theta\left[\alpha^2-1\right] = 0$$

$$\Rightarrow \quad \alpha^2 - 1 = 0 \quad \text{or} \quad \boxed{\alpha = \pm 1}$$

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$$\alpha = \pm 1 \implies a_1 \frac{\sin\theta}{r} + a_2 r \sin\theta$$

$$\Downarrow$$

$$\psi = a_2 r \sin\theta + a_1 \frac{\sin\theta}{r}$$

$$\therefore \psi = a_2 r \sin\theta + a_1 \frac{\sin\theta}{r} \rightarrow \text{(ii)}.$$

$$U_x = \sqrt{V_r^2 + V_\theta^2}$$

$$V_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$V_\theta = - \frac{\partial \psi}{\partial r}$$

$$V_r = \frac{1}{r}\left(\frac{a_1 \cos\theta}{r} + a_2 r \cos\theta \right)$$

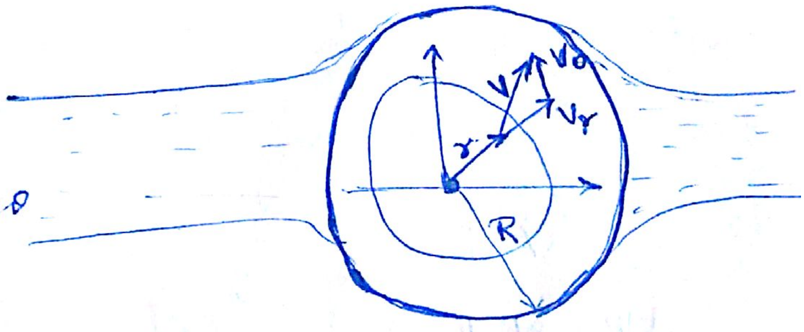
$$\boxed{V_r = \frac{a_1 \cos\theta}{r^2} + a_2 \cos\theta}$$

$$V_\theta = -\left(-\frac{a_1 \sin\theta}{r^2} + a_2 \sin\theta \right)$$

$$\boxed{V_\theta = \frac{a_1 \sin\theta}{r^2} - a_2 \sin\theta}$$

a)

Now as $r \rightarrow \infty$



$$V_r = a_2 \cos\theta, \quad V_\theta = -a_2 \sin\theta$$

$$V = U_x = \sqrt{V_r^2 + V_\theta^2}$$

$$= \sqrt{a_2^2 \cos^2\theta + a_2^2 \sin^2\theta}$$

$$\boxed{U_x = a_2}$$

$(\sin^2\theta + \cos^2\theta = 1)$

Now Consider,

at $r = R$, $\theta = \pi$, $V_r = 0$, $V_\theta = 0$ $\left(\begin{array}{l}\cos\pi = -1 \\ \sin\pi = 0\end{array}\right)$

$$V_r = -\frac{a_1 \cos\pi}{R^2} + a_2 \cos\pi$$

$$V_r = \frac{-a_1}{R^2} - U_x$$

$$0 = \frac{-a_1}{R^2} - U_x$$

$$\Rightarrow \boxed{a_1 = -U_x R^2}$$

Sub a_1 & a_2 in (ii).

$$\therefore \boxed{\psi = -\frac{U_x R^2 \sin\theta}{r} + U_x r \sin\theta}$$

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = -U_x R^2 \frac{\cos\theta}{r^2} + U_x \cos\theta$$

$$V_r = U_x\left[1 - \frac{R^2}{r^2}\right]\cos\theta$$

$$V_\theta = \frac{-\partial\psi}{\partial r} = -U_x \frac{R^2}{r^2}\sin\theta - U_x \sin\theta$$

$$V_\theta = -U_x \sin\theta\left[1 + \frac{R^2}{r^2}\right]$$

At $r = R$

$V_r = 0$, $V_\theta = -2U_x \sin\theta$

Hence found the appropriate Boundary Conditions. & xpression in terms of stream function.

$$r \to \infty \left(\sqrt{\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)^2 + \left(-\frac{\partial\psi}{r}\right)^2} = U_x\right)$$

At $r = R$, $\theta = \pi$

$V_r = 0$, $V_\theta = 0$

$\psi = f(r)$ at $r = R$

c) Velocity field

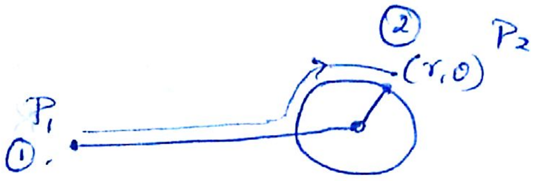
$$V = \sqrt{V_r^2 + V_\theta^2}$$

At $r = R$
$V_r = 0$
$V_\theta = -2U_x \sin\theta$

$$U = \sqrt{0 - (2U_x \sin\theta)^2}$$

$$\boxed{V = -2U_x \sin\theta}$$

(d)



Applying Bernoulli's theorem b/w ① & ②

(V_2)

$$P_1 + \tfrac{1}{2}\rho \underset{(V_1)}{V_x^2} = P_2 + (-2U_x \sin\theta)^2 \tfrac{1}{2}\rho$$

$$P_1 + \tfrac{1}{2}\rho U^2 = P_2 + \frac{4U^2 \sin^2\theta\, \rho}{2}$$

$$P_1 - P_2 = \frac{\cancel{4}^{2}\, U^2 \sin^2\theta}{\cancel{2}}\rho - \tfrac{1}{2}\rho U^2$$

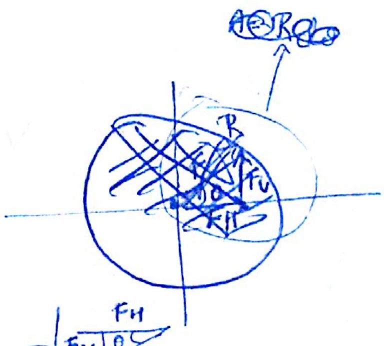
$$P_1 - P_2 = U^2\rho\left[2\sin^2\theta - \tfrac{1}{2}\right]$$

$\left(\sin^2\theta = \frac{1 - \cos 2\theta}{2}\right)$

$$P_1 - P_2 = U^2\rho\left[\frac{2(1 - \cos 2\theta)}{2} - \tfrac{1}{2}\right]$$

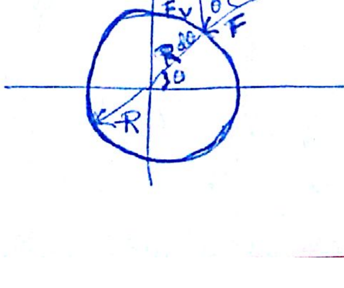
* Pressure field $\quad P = \boxed{P_2 - P_1 = U^2\rho\left[\cos 2\theta - \tfrac{1}{2}\right]}$

(e) compute Net Force acting on cylinder :-



$$F_H = \int_0^{2\pi} PR\, d\theta \cos\theta$$

$$F_V = \int_0^{2\pi} PR\, d\theta \sin\theta$$



$$F_H = \int_0^{2\pi} U^2 \rho \left[\cos 2\theta - \tfrac{1}{2}\right] R \, d\theta \, \cos\theta$$

$$= U^2 \rho R \left[\underbrace{\int_0^{2\pi} \cos 2\theta \cos\theta \, d\theta}_{\text{Integrate by parts}} - \tfrac{1}{2} \int_0^{2\pi} \cos\theta \, d\theta \right]$$

$$\int_0^{2\pi} \cos\theta \cos 2\theta \, d\theta = \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \int_0^{2\pi} \frac{\sin\theta \sin 2\theta}{2} \, d\theta$$

$$= \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \left[\sin\theta \int_0^{2\pi} \sin 2\theta \, d\theta - \int_0^{2\pi} \frac{\cos 2\theta \cos\theta \, d\theta}{2} \right.$$

$$\text{Let } \int_0^{2\pi} \cos 2\theta \cos\theta = P$$

$$= \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \left[\sin\theta \int_0^{2\pi} \sin 2\theta \, d\theta - \frac{P}{2} \right.$$

$$\frac{P}{2} = \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \sin\theta \int_0^{2\pi} \sin 2\theta \, d\theta$$

$$\frac{P}{2} = \left[\frac{\cos\theta \cancelto{0}{\sin 2\theta}}{2} \right]_0^{2\pi} - \left[\frac{\cancelto{0}{\sin\theta} \cos 2\theta}{2} \right]_0^{2\pi} \quad (\sin 2\pi = 0)$$

$$\boxed{P = 0}$$

$$\therefore F_H = U^2 \rho R \left[0 - \tfrac{1}{2} \left[\cancelto{0}{\sin\theta} \right]_0^{2\pi} \right.$$

$$\boxed{F_H = 0}$$

$$F_V = \int_0^{2\pi} p R \sin\theta \, d\theta = \int_{-\pi}^{\pi} U^2 \rho R \left[\cos 2\theta - \tfrac{1}{2}\right] \sin\theta \, d\theta$$

$$\Rightarrow U^2 \rho R (-) \int_{\pi}^{0} \left[\cos(-2\theta) - \tfrac{1}{2}\right] \cdot \sin(-\theta) \, d\theta \qquad \begin{bmatrix} \int \sin(-\theta) = \int -\sin\theta \\ \int \cos(-\theta) = \int \cos\theta \\ \text{(odd function property)} \end{bmatrix}$$

$$\downarrow$$

$$+ \int_0^{\pi} \left[\cos 2\theta - \tfrac{1}{2}\right] \cdot \sin\theta \, d\theta$$

$$= U^2 \rho R \left[+ \int_0^{\pi} \left[\cos 2\theta - \tfrac{1}{2}\right](-\sin\theta) + \int_0^{\pi} \left[\cos 2\theta - \tfrac{1}{2}\right] \sin\theta \, d\theta \right]$$

$$= U^2 \rho R \left[- \int_0^{\pi} \left[\cos\theta - \tfrac{1}{2}\right] \sin\theta \; \cancel{+ \int_0^{\pi} \left[\cos 2\theta - \tfrac{1}{2}\right] \sin\theta \, d\theta} \right] \rightarrow 0$$

$(\sin \pi = 0)$

$$\boxed{F_V = 0}$$

$\therefore$ Net force acting on cylinder,

$$F_{net} = \sqrt{F_H^2 + F_V^2} = 0$$

$$\boxed{F_{net} = 0}$$

Comments :-

∴ This Result makes sense according to 'De Alembert's Paradox' which states that drag force ($F = 0$) on a body ~~road~~ moving with constant velocity relative to fluid.