Hypotheses:

1. steady state $\frac{du}{dt}=0$
2. No body-forces $\rho X = 0$
3. Newtonian Incompressible fluid
4. viscous fluid
5. $\frac{\partial p}{\partial x} \to cte$
6. $\phi = \frac{R_2}{R_1}$

cylindrical coordinates:

Eq. motion:

(1) $\cancel{\frac{\partial \rho}{\partial t}} + \cancel{\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)} + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)} + \frac{\partial}{\partial x}(\rho v_x) = 0 \Rightarrow \frac{\partial}{\partial x}(\rho v_x) = 0^*$

r-momentum:

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + \cancel{v_r\frac{\partial v_r}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta^2}{r}} + \cancel{v_x\frac{\partial v_r}{\partial x}}\right) = -\frac{\partial p}{\partial r} + \mu\left(\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 v_r}{\partial x^2}}\right) + \cancel{\rho b_r}$$

θ-momentum:

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r v_\theta}{r}} + \cancel{v_x\frac{\partial v_\theta}{\partial x}}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial x^2}}\right) + \cancel{\rho b_\theta}$$

$$\frac{1}{r}\left(\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}\right)\mu$$

x-momentum:

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + \cancel{v_r\frac{\partial v_x}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_x}{\partial \theta}} + \cancel{v_x\frac{\partial v_x}{\partial x}}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_x}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_x}{\partial x^2}}\right) + \cancel{\rho b_x}$$

$v = [\,\cancel{v_r(r,\theta,x)}\quad \cancel{v_\theta(r,\theta,x)}\quad v_x(r,\cancel{\theta},\cancel{x})\,]^* \Rightarrow v_x(r);\ p = p(r,\theta,x)$

properties don't change in θ

$-\frac{\partial p}{\partial r} = 0 \Rightarrow -\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \Rightarrow p = p(x)$

$A = \frac{\partial p}{\partial x}$

$$-\frac{\partial p}{\partial x} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right)\right) = 0 \Rightarrow \frac{\partial p}{\partial x} = \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right)\right) \Rightarrow \frac{1}{\mu}\frac{\partial p}{\partial x} = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right)\right) \Rightarrow \frac{A}{\mu} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) = \frac{Ar}{\mu} \Rightarrow \int \partial\left(r\frac{\partial v_x}{\partial r}\right) = \int \frac{Ar}{\mu}\,dr \Rightarrow r\frac{\partial v_x}{\partial r} = \frac{Ar^2}{2\mu} + B \Rightarrow \frac{\partial v_x}{\partial r} = \frac{Ar}{2\mu} + \frac{B}{r} \Rightarrow \int \partial v_x = \int\left(\frac{Ar}{2\mu} + \frac{B}{r}\right)dr \Rightarrow$$

$$\Rightarrow v_x = \frac{Ar^2}{4\mu} + B\ln r + C \Rightarrow v_x(r) = \frac{Ar^2}{4\mu} + B\ln r + C$$

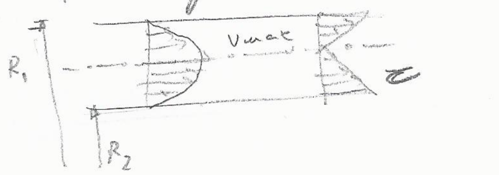
BC's

$v_x(r)\big|_{r=R_2} = 0 \Rightarrow v_x(r)\big|_{r=R_2} = \frac{AR_2^2}{4\mu} + B\ln R_2 + C = 0$

$v_x(r)\big|_{r=R_1} = 0 \Rightarrow v_x(r)\big|_{r=R_1} = \frac{AR_1^2}{4\mu} + B\ln R_1 + C = 0$

$A = \frac{\partial p^*}{\partial x}$ $\quad v_{max}(r)\Big|_{r=\frac{R_1-R_2}{2}}$

flow forming paraboloid of revolution



$$B = -\frac{a(R_1^2 - R_2^2)}{4\mu(\ln R_1 - \ln R_2)\ln\frac{R_1}{R_2}} \qquad C = \frac{a(R_1^2\ln R_2 - R_2^2\ln R_1)}{4\mu(\ln R_1 - \ln R_2)\ln\frac{R_1}{R_2}}$$

$$v_x(r) = \frac{\partial p^*}{\partial x}\frac{r^2}{4\mu} - \frac{A(R_1^2 - R_2^2)}{4\mu(\ln R_1 - \ln R_2)}\ln r + \frac{a(R_1^2\ln R_2 - R_2^2\ln R_1)}{4\mu(\ln R_1 - \ln R_2)} \Rightarrow$$

$$\Rightarrow v_x(r) = \frac{\partial p^*}{\partial x}\frac{r^2}{4\mu} - \frac{\partial p^*}{\partial x}\frac{(R_1^2 - R_2^2)}{4\mu(\ln R_1 - \ln R_2)}\ln r + \frac{\partial p^*}{\partial x}\frac{(R_1^2\ln R_2 - R_2^2\ln R_1)}{4\mu(\ln R_1 - \ln R_2)}.$$

$$\frac{AR_2^2}{4\mu} + B\ln R_2 + C = 0 \Rightarrow \frac{AR_2^2}{4\mu} + B\ln R_2 - \left(\frac{AR_1^2}{4\mu} + B\ln R_1\right) = 0$$

$$\frac{AR_1^2}{4\mu} + B\ln R_1 + C = 0 \Rightarrow -\left(\frac{AR_1^2}{4\mu} + B\ln R_1\right) = C$$

$$\underbrace{\frac{AR_2^2}{4\mu} - \frac{AR_1^2}{4\mu}}_{(R_2^2 - R_1^2)\frac{A}{4\mu}} + \underbrace{B\ln R_2 - B\ln R_1}_{B\ln\frac{R_2}{R_1}} = 0$$

$$B = (R_1^2 - R_2^2)\frac{A}{4\mu\ln\frac{R_2}{R_1}}$$

$$\cancel{AR}\,C = -\left(\frac{AR_1^2}{4\mu} + \frac{(R_1^2 - R_2^2)A}{4\mu\ln\frac{R_2}{R_1}}\right)$$

$$C = -\frac{AR_1^2}{4\mu} + \frac{(R_2^2 - R_1^2)A\ln R_1}{4\mu\ln\frac{R_2}{R_1}} = \frac{A}{4\mu}\left(-R_1^2 + \frac{(R_2^2 - R_1^2)\ln R_1}{\ln\frac{R_2}{R_1}}\right)$$

$$u(r) = \frac{Ar^2}{4\mu} + B\ln r + C = \frac{Ar^2}{4\mu} + \frac{(R_1^2 - R_2^2)A}{4\mu\ln\frac{R_2}{R_1}}\ln r + \frac{A}{4\mu}\left(-R_1^2 + \frac{(R_2^2 - R_1^2)\ln R_1}{\ln R_2}\right)$$

$$= \frac{Ar^2}{4\mu} - \frac{R_1^2}{4\mu} + \frac{A}{4\mu}\left(\frac{(R_1^2 - R_2^2)\ln r}{\ln\frac{R_2}{R_1}} + \frac{(R_2^2 - R_1^2)\ln R_1}{\ln\frac{R_2}{R_1}}\right) = \frac{A}{4\mu}(R_1^2 - r^2) + \frac{A}{4\mu}\left(\frac{(R_2^2 - R_1^2)\ln r}{\ln\frac{R_2}{R_1}} - \frac{(R_2^2 - R_1^2)\ln R_1}{\ln\frac{R_2}{R_1}}\right)$$

$$= \frac{A}{4\mu}(R_1^2 - r^2) + \frac{A}{4\mu}\left(\frac{(R_2^2 - R_1^2)\ln r/R_1}{\ln\frac{R_2}{R_1}}\right) = \frac{A}{4\mu}\left[(R_1^2 - r^2) + \frac{(R_2^2 - R_1^2)\ln r/R_1}{\ln\frac{R_2}{R_1}}\right]$$

$$Q = \int_{R_2}^{R_1} u\,2\pi(R_1 - R_2)\,dr = \int_{R_2}^{R_1} \frac{A}{4\mu}\left[(R_1^2 - r^2) + \frac{(R_2^2 - R_1^2)\ln r/R_1}{\ln\frac{R_2}{R_1}}\right]2\pi r\,dr$$

$$Q = \frac{A\pi}{8\mu}\left[\frac{(R_1^2 - R_2^2)^2}{\ln\frac{R_2}{R_1}} + (R_1^2 + R_2^2)(R_1^2 - R_2^2)\right] = \frac{A\pi}{8\mu}\left[\frac{(R_1^2 - R_2^2)^2}{\ln\frac{R_2}{R_1}} + R_1^4 - R_2^4\right]$$

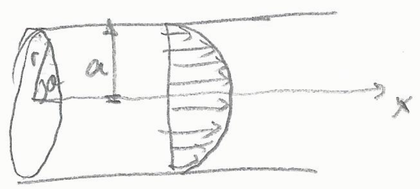
if $R_2 = r$, $R_1 = 0$

$$Q = \frac{A\pi}{8\mu}\left[\underbrace{\frac{-r^4}{\ln\frac{r}{0} \to -\infty}}_{=0} - r^4\right] = -\frac{A\pi r^4}{8\mu} \quad \text{if } A = \frac{\partial p^*}{\partial x} = -\frac{\partial p^*}{\partial x}\frac{\pi r^4}{8\mu}$$

if $\frac{\partial p^*}{\partial x} = -\rho g \Rightarrow \frac{\pi r^4 \rho g}{8\mu} = Q \Rightarrow$ if $\frac{\mu}{\rho} = \nu \Rightarrow \boxed{Q = \frac{\pi r^4 g}{8\nu}}$ $\Rightarrow$ Yes, it reduces to the original formula for Poiseuille flow in a circular pipe of radius R_1

if $r = R_1 \Rightarrow Q = \frac{\pi R_1^4 g}{8\nu}$

$$Q = -\frac{1}{8}\frac{\pi a^4}{\mu}\frac{\partial p}{\partial t} \Rightarrow \text{being } a$$



Von Karman – Pohlhausen:

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2 \Rightarrow \text{ if } \eta = \frac{y}{\delta} \quad \frac{u}{U} = a + b\eta + c\eta^2 \Rightarrow \eta = \frac{y}{\delta} \Rightarrow dy = \delta\, d\eta$$

BC's

$$u=0,\ y=0 \Rightarrow \eta = 0,\ \frac{u}{U} = 0 \qquad ① \qquad 0 = a$$

$$u=U,\ y=\delta \Rightarrow \eta = 1,\ \frac{u}{U} = 1 \qquad ② \qquad 1 = b+c$$

$$\frac{\partial u}{\partial y} = 0,\ y=\delta \Rightarrow \eta = 1,\ \frac{\partial (u/U)}{\partial \eta} = 0 \qquad ③ \qquad 0 = b + 2c\eta$$

$$\eta = 1 \Rightarrow 0 = b + 2c$$

$$\left.\begin{aligned} 1 &= b+c \\ 0 &= b+2c \end{aligned}\right\}$$

$$b = 1-c$$

$$0 = 1 - c + 2c \Rightarrow 0 = 1 + c \Rightarrow c = -1$$

$$b = 1 - c = 1 + 1 = 2$$

$$\left.\begin{aligned} a &= 0 \\ b &= 2 \\ c &= -1 \end{aligned}\right\} \frac{u}{U} = 2\eta - \eta^2$$

governing eq. ⇒ Von Karman moment integral eq.

$$U^2 \frac{\partial \delta''}{\partial x} + (2\delta'' + \delta')U\frac{\partial U}{\partial x} = \frac{\tau_w}{\rho} \Rightarrow \frac{\partial P}{\partial x} = 0 \Rightarrow U\frac{\partial U}{\partial x} = 0 \Rightarrow \frac{\partial \delta''}{\partial x} = \frac{\tau_w}{\rho U^2} \quad ①$$

$$\delta'' = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \Leftarrow \text{momentum thickness} \Rightarrow \delta'' = \int_0^\eta (2\eta - \eta^2)(1 - (2\eta - \eta^2))\, d\eta$$

$$\delta'' = \delta \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta = \delta \int_0^1 (2\eta - 4\eta^2 + 2\eta^3 - \eta^2 + 2\eta^3 - \eta^4)\, d\eta =$$

$$= \delta \left[\frac{2\eta^2}{2} - \frac{4\eta^3}{3} + \frac{2\eta^4}{4} - \frac{\eta^3}{3} + \frac{2\eta^4}{4} - \frac{\eta^5}{5}\right]_0^1 = \delta\left(1 - \frac{4}{3} + \frac{2}{4} - \frac{1}{3} + \frac{2}{4} - \frac{1}{5}\right) = \delta\left(\cancel{-1} + \cancel{1} + 1 - \frac{1}{5}\right) = \frac{2}{15}$$

$$\delta'' = \frac{2}{15}\delta$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \Rightarrow \tau_w = \mu\left[\frac{\partial}{\delta \partial \eta}\left(U(2\eta - \eta^2)\right)\right]_{\eta=0} = \frac{2\mu U}{\delta}$$

$$① \quad \frac{\partial \delta''}{\partial x} = \frac{\tau_w}{\rho U^2} \Rightarrow \frac{\partial \delta}{\partial x}\frac{2}{15} = \frac{2\mu U}{\delta \rho U^2} \Rightarrow \nu = \frac{\mu}{\rho} \Rightarrow \int \partial\delta\, \delta = \int \frac{30\mu}{2\rho U}\partial x \Rightarrow \frac{\delta^2}{2} = \frac{30\mu x}{2\rho U} + A$$

at leading edge

$$x=0,\ \delta = 0 \Rightarrow A = 0 \Rightarrow \frac{\delta^2}{2} = \frac{30\mu x}{2\rho U} \Rightarrow \delta^2 = \frac{30\nu x}{U} \Rightarrow \delta^2 = \frac{30\nu x}{U} = \sqrt{30}\sqrt{\frac{\nu x}{U}}$$

$$\left[\delta = \frac{\sqrt{5}\, x}{\sqrt{Re_x}} \approx \frac{2{,}23x}{\sqrt{Re_x}}\right]$$

$$\text{Blasius} \Rightarrow \delta \to u = 0{,}99U \Rightarrow \frac{u}{U} = 0{,}99 \to \eta = 5.0 \Rightarrow 5.0 = \delta\sqrt{\frac{U}{\nu x}} \Rightarrow \delta = \frac{5.0}{\sqrt{\frac{U}{\nu x}}} = \frac{5.0x}{\sqrt{\frac{Ux^2}{\nu x}}} =$$

$$= \frac{5.0x}{\sqrt{\frac{Ux}{\nu}}} = \frac{5.0x}{\sqrt{Re_x}}$$

VK-P approximation

Blasius exact solution

$$\delta_{VK} = \frac{5{,}477x}{\sqrt{Re_x}} \quad \delta_B = \frac{5.0x}{\sqrt{Re_x}} \Rightarrow \frac{\frac{5{,}477x}{\sqrt{Re_x}} - \frac{5.0x}{\sqrt{Re_x}}}{\frac{5{,}0x}{\sqrt{Re_x}}} = 0{,}09 \simeq 10\%\ \text{Error}$$