Advanced Fluid Mechanics

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Homework 4

1. a) To find velocity components we can rewrite stream function:

$\psi(r,\theta) = U \cdot r^2 \sin(2\theta) = U \cdot r^2 \cdot 2 \cdot \sin\theta \cdot \cos\theta$

$r\sin\theta = y$; $r\cos\theta = x$

$\psi(x,y) = 2 \cdot x \cdot y \cdot U$

The velocity components:

$u = \frac{\partial\psi}{\partial y} = 2 \cdot U \cdot x$

$v = -\frac{\partial\psi}{\partial x} = -2 \cdot U \cdot y$

It verifies boundary conditions as:

at $y=0$ $v=0$; at $x=0$ $u=0$

To find pressure distribution we will express pressure in stagnation point:

$p_0 = p + \frac{\rho V^2}{2}$

$V = \sqrt{u^2+v^2} = 2U\sqrt{x^2+y^2}$

$p = p_0 - \rho \cdot 2U^2(x^2+y^2)$

b) The former velocity and pressure verify Navier-Stokes equations, because viscous-shear term is identically zero for potential flow:

$\nabla^2 V = \nabla^2(\nabla\varphi) = 0$

But no-slip boundary condition is not satisfied as:

$u \neq 0$ at $y=0$

c) if the x-component of velocity is taken as

$u = 2Ux f'(y)$

$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -2Uf'(y)$

then the second component

$v = -2Uf(y)$

The appropriate boundary condition for $f(y)$ is

if $y \to \infty$ then $f(y) \to y$

d) To obtain pressure distribution we will use y-momentum equation:

$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$

$4U^2 f = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2U\nu f$

After integration of obtained expression with respect to y we get:

$p = -2\rho U^2 f^2 - 2\rho U \nu f' + f_2(x)$

To find $f_2(x)$ we will use the property that for high values of y the potential flow should be recovered.

$f_2(y) = p_0 - 2\rho U^2 x^2 - 2\rho U\nu$

Substituting this value into the pressure distribution equation:

$$P = p_0 - 2\rho U^2 f^2 + 2\rho U\nu(-f'+1) - 2\rho U^2 x^2$$

e) The x-momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

From the pressure distribution equation we can find $\frac{\partial p}{\partial x}$ term:

$$\frac{\partial p}{\partial x} = -4\rho U^2 x$$

The x-momentum equation will be as follows:

$$4U^2 x (f')^2 - 4U^2 x \cdot f \cdot f'' = 4U^2 x + 2U\nu x f'''$$

$$\frac{\nu}{2U} f''' + f \cdot f'' - (f')^2 + 1 = 0$$

To show that above-mentioned boundary conditiones are valid we will evaluate equation at $y \to \infty$:

$f(y) \to y$; $f'(y) \to 1$; $f''(y), f'''(y) \to 0$

$0 + 0 - 1 + 1 = 0$

Therefore boundary conditions are satisfied by the differential equation.

2. To find constants we will apply boundary conditions to the equation:

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2$$

$u=0$, $y=0$ gives $a=0$

$u=U$ at $y=\delta$ gives $b+c=1$

$\frac{\partial u}{\partial y}=0$ at $y=\delta$ gives $\frac{\partial u}{\partial y} = \frac{b}{\delta} + \frac{2cy}{\delta^2} = 0$; $b+2c=0$

$$\begin{cases} b+c=1 \\ b+2c=0 \end{cases}$$

$$\begin{cases} c=-1 \\ b=2 \end{cases}$$

$$\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

Now we can evaluate both sides of the equation:

$$\frac{d}{dx}\int_0^\delta (U-u)u\,dy = \frac{\tau_0}{\rho}$$

$$\frac{\tau_0}{\rho} = \nu\left(\frac{\partial u}{\partial y}\right)_0 = \frac{2U\nu}{\delta}$$

$$\int_0^\delta (U-u)u\,dy = \int_0^\delta \left(U - 2U\frac{y}{\delta} + U\left(\frac{y}{\delta}\right)^2\right)\left(2U\frac{y}{\delta} - U\left(\frac{y}{\delta}\right)^2\right)dy = \int_0^\delta \Big(2U^2\frac{y}{\delta} - U^2\frac{y^2}{\delta^2} - 4U^2\frac{y^2}{\delta^2} + 2U^2\frac{y^3}{\delta^3} + 2U^2\frac{y^3}{\delta^3} - U^2\frac{y^4}{\delta^4}\Big)dy = U^2\delta - \frac{5}{3}U^2\delta + U^2\delta - \frac{1}{5}U^2\delta = \frac{15U^2\delta - 25U^2\delta + 15U^2\delta - 3U^2\delta}{15} = \frac{2U^2\delta}{15}$$

Substituting into the equation:

$$\frac{2}{15} U^2 \frac{d\delta}{dx} = \frac{2 U \nu}{\delta}$$

$$\int \delta \, d\delta = \frac{15 \nu}{U}$$

$$\frac{\delta^2}{2} = \frac{15 \nu}{U}$$

$$\delta = 5{,}48 \sqrt{\frac{\nu x}{U}}$$

The friction factor:

$$C_f = \frac{\tau_0}{(1/2)\rho U^2} = \frac{2 U \nu / \delta}{\frac{1}{2} U^2} = \frac{0{,}73}{\sqrt{Re_x}}$$

It overestimates exact Blasius solution ($\frac{0{,}664}{\sqrt{Re_x}}$) and is less accurate than approximation with cubic profile ($\frac{0{,}646}{\sqrt{Re_x}}$).