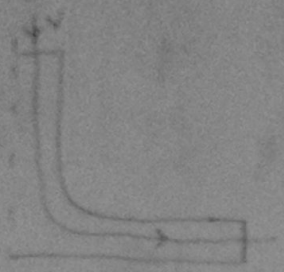
Home Work 2

Arnab Samaddar Chaudhuri

1)



Apply mass conservation

$$\frac{DM}{Dt} = \frac{\partial}{\partial t}\int_V \rho\, dV + \int_S \rho v \cdot n\, dS = 0$$

for steady state

$$\int_S \rho v \cdot n\, dS = 0$$

Assuming constant density $\rho$

$$\Rightarrow -\rho v \pi a^2 + \rho v_r \cdot 2\pi r h = 0$$

Using $V_r = C_2/r$ for $r > a$, we have

$$\Rightarrow \rho v \pi a^2 = \rho \frac{C_2}{r} 2\pi r h \quad \Rightarrow C_2 = \frac{Va^2}{2h}$$

$$\therefore V_r = \frac{Va^2}{2rh} \quad \text{for } r > a$$

Since $V_r = C_1 r$ for $r < a$, we have

$$V_r = \frac{C_2}{r} = C_1 r \quad \text{at } r = a$$

$$\Rightarrow \frac{Va^2}{2ah} = C_1 a \quad \Rightarrow C_1 = \frac{V}{2h}$$

$$\boxed{\begin{aligned} V_r &= \frac{Vr}{2h} \quad \text{for } r \le a \\ V_r &= \frac{Va^2}{2rh} \quad \text{for } r \ge a \end{aligned}}$$

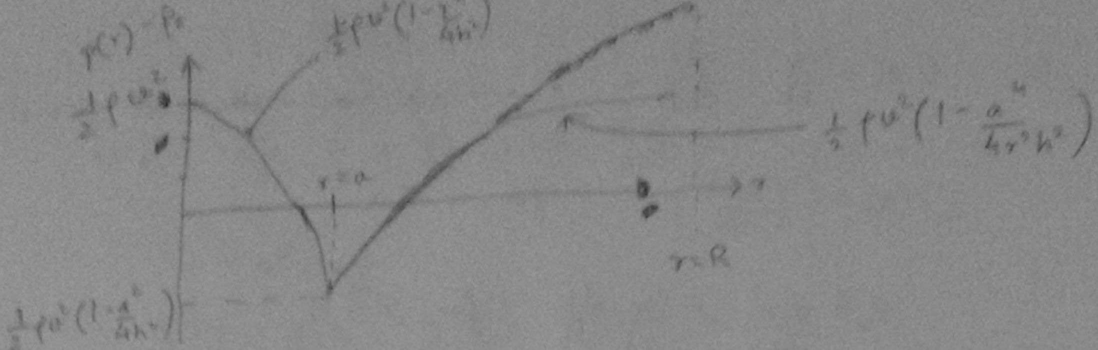
b) Applying bernoullis eq

$$\frac{P_0}{\rho} + \frac{V^2}{2} = \frac{p(r)}{\rho} + \frac{V_r^2}{2}$$

$$\Rightarrow p(r) - P_0 = \frac{1}{2}\rho v^2 - \frac{1}{2}\rho v_r^2$$

$$= \frac{1}{2}\rho v^2\left(1 - \frac{r^2}{4h^2}\right) \quad \text{for } r \le a$$

$$= \frac{1}{2}\rho v^2\left(1 - \left(\frac{a^2}{2rh}\right)^2\right) \quad \text{for } r \ge a$$




$$F_p = \int_0^R (p(r)-p_0)\,2\pi r\,dr$$

$$= \int_0^a \frac{1}{2}\rho v^2\left(1-\frac{r^2}{4h^2}\right)2\pi r\,dr + \int_a^R \frac{1}{2}\rho v^2\left(1-\frac{a^4}{4h^2r^2}\right)2\pi r\,dr$$

$$= \frac{1}{2}\rho v^2 \pi R^2 - \frac{1}{2}\rho v^2 2\pi\left[\int_0^a \frac{r^2}{4h^2}\,r\,dr + \int_a^R \frac{a^4}{4h^2}\frac{dr}{r}\right]$$

$$= \frac{1}{2}\rho v^2 \pi R^2 - \frac{1}{2}\rho v^2 2\pi\left[\frac{a^4}{16h^2} + \frac{a^4}{4h^2}\ln\frac{R}{a}\right]$$

$$= \frac{1}{2}\pi\rho v^2\left(R^2-\frac{a^4}{8h^2}\right) - \rho v^2\pi\frac{a^4}{4h^2}\ln\left(\frac{R}{a}\right)$$

$$\Rightarrow F_p = \frac{1}{2}\pi\rho v^2\left(R^2-\frac{a^4}{8h^2}\right) + \pi\rho v^2\frac{a^4}{4h^2}\ln\frac{a}{R}$$

This force can be negative when $R \gg a$ and $\frac{a^2}{h} \gg R$.

$a = 0.01\,m$, $R = 0.05\,m$, $h = 0.001\,m$, $m = 0.01\,kg$

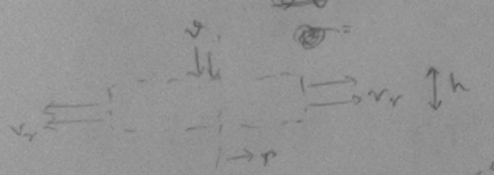
$\rho = 1$ ; $F_{down} + weight_{(disk)} = 0 \Rightarrow F_{up} = weight$

Plugging the above values,

$$10^{-3}\times 9.81 + \frac{\pi}{2}\times 1\times v^2\left[0.05^2 - \frac{0.01^4}{8\times(0.001)^2} + \frac{0.01^4}{4\times 0.001^2}\ln\frac{0.01}{0.05}\right]$$

$\Rightarrow v = 1.7\ m/s$

d)




Apply reynold's transport theorem in the control volume

$$\frac{d}{dt}\int \rho\,dV = \int_V \frac{\partial\rho}{\partial t}\,dV + \int \rho\,v\cdot n\,ds$$

$$\Rightarrow \frac{d}{dt}(\rho\pi r^2 h) = -\rho v\,\pi a^2 + \rho v_r\,2\pi r h$$

$$\Rightarrow r^2\frac{dh}{dt} = va^2 - 2v_r r h$$

$$\Rightarrow v_r = \frac{-r}{2h}\frac{dh}{dt} + \frac{a^2 v}{2hr} \quad \text{for } r > a$$

for $r < a$, we get

$$v_r = c_1\left(\frac{dh}{dt}, h\right) r$$

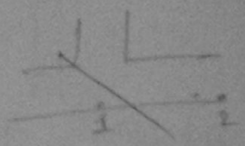
$$c_1\left(\frac{dh}{dt}, h\right) r\Big|_{r=a} = -\frac{a\,dh}{dt} + \frac{av}{2h}$$

$$\Rightarrow c_1 = \frac{v}{2h} - \frac{dh}{2h\,dt}$$

$$\therefore v_r = \left(\frac{v-h}{2h}\right) r \quad \text{for } r < a$$

$$v_r = \frac{v a^2}{2hr} - \frac{r}{2h}\frac{dh}{dt} \quad \text{for } r \geq a$$

Appling bernoullis theorem




$$\int_1^2 \frac{\partial v}{\partial t}\, ds + \left[\frac{v^2}{2} + \frac{p}{\rho}\right]_1^2 = 0$$

$$\int_1^2 \frac{\partial v}{\partial t}\, ds + \frac{v_2^2}{2} + \frac{P_2}{\rho} - \frac{v_1^2}{2} - \frac{P_1}{\rho} = 0$$

$$\int_1^2 \frac{\partial v}{\partial t}\, ds = \int_0^a -\frac{d^2h}{dt^2}\, r\, dr + \int_a^R \left[\frac{r}{2h}\frac{d^2h}{dt^2} - \frac{1}{h}\frac{dh}{dt}\left(\frac{a^2 v - r^2\frac{dh}{dt}}{2r}\right)\right] r$$

$$= P_1 + \cancel{\frac{v_1^2}{2}} - P_2 - \frac{v_2^2}{2}$$

$$= P_1 - P_2 - \left(\frac{a^2 v - r^2\frac{dh}{dt}}{2h}\right)$$

Applying bernoullis b/w $r = 0$ & $r = R$

$$\int_0^R \frac{\partial v}{\partial t}\, ds + P_R + \frac{1}{2}\rho v_R^2 - P_2 - \frac{1}{2}\rho v_2^2 = 0$$

$$P_0 = P_R$$

$$P_2 = P_1 + \frac{1}{2}\rho v^2 \qquad [P_1 = P_0]$$

$$= P_0 + \frac{1}{2}\rho v^2$$

$$P_R - P_2 = -\frac{1}{2}\rho v^2 \quad (P_0 = P_R)$$

$$v_r = \frac{v a^2}{2Rh} - \frac{R}{2h}\left(\frac{dh}{dt}\right)$$

$$\cancel{\int_0^a \rho\left[-\frac{v_r}{2h^2}\frac{dh}{dt}\right.} \quad \int_0^a \frac{\partial v}{\partial t}\, ds + \int_a^R \frac{\partial v}{\partial t}\, ds - \frac{1}{2}\rho v^2 + \frac{1}{2}\rho\left[\frac{va^2}{2Rh} - \frac{R}{2h}\frac{dh}{dt}\right]^2 = 0$$

$$\int_0^a \rho\left[-\frac{v_r}{2h^2}\frac{dh}{dt} + \frac{r}{2h^2}\left(\frac{dh}{dt}\right)^2 - \frac{r}{2h}\frac{d^2h}{dt^2}\right] dr + \int_a^R \rho\left[-\frac{va^2}{2rh^2}\frac{dh}{dt} + \frac{r}{2h^2}\left(\frac{dh}{dt}\right)^2 - \frac{r}{2h}\frac{d^2h}{dt^2}\right] dr - \frac{1}{2}\rho v^2 + \frac{1}{2}\rho\left[\frac{va^2}{2Rh} - \frac{R}{2h}\frac{dh}{dt}\right]^2 = 0$$

1) Laplacian in polar coordinates

$$\nabla^2 a = \frac{\partial^2 a}{\partial r^2} + \frac{1}{r}\frac{\partial a}{\partial r} + \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2}$$

$$\psi = r^\alpha \sin\theta$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \quad -(1)$$

$$\frac{\partial \psi}{\partial r} = \alpha r^{\alpha-1}\sin\theta, \quad \frac{\partial^2 \psi}{\partial r^2} = (\alpha^2-\alpha) r^{\alpha-2}\sin\theta$$

$$\frac{\partial \psi}{\partial \theta} = r^\alpha \cos\theta, \quad \frac{\partial^2 \psi}{\partial \theta^2} = -r^\alpha \sin\theta$$

Substituting in (1)

$$\nabla^2 \psi = (\alpha^2-\alpha) r^{\alpha-2}\sin\theta + \frac{1}{r}\alpha r^{\alpha-1}\sin\theta + \frac{1}{r^2}\cdot r^\alpha \sin\theta$$

$$= (\alpha^2-\alpha) r^{\alpha-2}\sin\theta + \alpha r^{\alpha-2}\sin\theta + r^{\alpha-2}\sin\theta$$

$$\nabla^2 \psi = r^{\alpha-2}\sin\theta\left[\alpha^2-\alpha+\alpha+1\right]$$

$$\Delta\psi = r^{\alpha-2}\sin\theta\,(\alpha^2-1)$$

The plane flow is irrotational

$$\Delta\psi = 0$$

$$r^{\alpha-2}\sin\theta\,(\alpha^2-1) = 0$$

$$\Rightarrow \alpha^2-1=0 \quad \Rightarrow \alpha = \pm 1$$

$$\alpha = \pm 1 \Rightarrow a_1 \frac{\sin\theta}{r} + a_2 r\sin\theta$$

$$\psi = a_2 r\sin\theta + a_1 \frac{\sin\theta}{r} \quad -(2)$$

$$\psi = a_2 r\sin\theta + \frac{a_1 \sin\theta}{r} \quad -(2)$$

$$v_\infty = \sqrt{V_r^2 + V_\theta^2}, \quad V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad V_\theta = -\frac{\partial \psi}{\partial r}$$

$$V_r = \frac{1}{r}\left(a_1 \frac{\cos\theta}{r} + a_2 r\cos\theta\right)$$

$$= \frac{a_1 \cos\theta}{r^2} + a_2\cos\theta$$

$$V_\theta = -\left(-\frac{a_1 \sin\theta}{r^2} + a_2\sin\theta\right)$$

$$= a_1 \frac{\sin\theta}{r^2} - a_2\sin\theta$$

a) Now as $r \to \infty$

$$V_r = a_2\cos\theta, \quad V_\theta = -a_2\sin\theta$$

$$V = v_\infty = \sqrt{V_r^2 + V_\theta^2}$$

$$= \sqrt{a_2^2\cos^2\theta + a_2^2\sin^2\theta}$$

$$u_\infty = a_2$$

Scanned by CamScanner

Now at $r = R$, $\theta = \pi$ $\quad V_r = 0$, $V_\theta = 0$

$$V_r = -\frac{a_1 \cos\pi}{R^2} + a_2 \cos\pi$$

$$0 = -\frac{a_1}{R^2} - U_x$$

$$\Rightarrow a_1 = -U_x R^2$$

Putting values of $a_1$ & $a_2$ in (2)

$$\psi = -U_x \frac{R^2 \sin\theta}{r} + U_x r \sin\theta$$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -U_x \frac{R^2 \cos\theta}{r^2} + U_x \cos\theta$$

$$\Rightarrow V_r = U_x\left(1 - \frac{R^2}{r^2}\right)\cos\theta$$

$$V_\theta = \frac{-\partial \psi}{\partial r} = -U_x \frac{R^2}{r^2}\sin\theta - U_x \sin\theta$$

$$V_\theta = -U_x \sin\theta \left(1 + \frac{R^2}{r^2}\right)$$

At $r = R$

$V_r = 0$, $V_\theta = 2U_x \sin\theta$

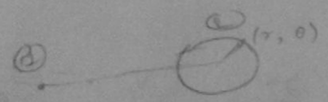
$\therefore$ The appropiate B.C.

when $r \to \infty$, $\quad U_x = \sqrt{\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)^2 + \left(\frac{\partial \psi}{r}\right)^2}$

c) velocity field $\quad V = \sqrt{V_r^2 + V_\theta^2}$ $\qquad$ At $r = R$, $V_r = 0$, $V_\theta = -2U_x \sin\theta$

$$= \sqrt{0 - (2U_x \sin\theta)^2}$$

$$= -2U_x \sin\theta$$




d) Applying bernoulli's theorem

$$\frac{P_1}{\rho} + \frac{1}{2}V_1^2 = \frac{P_2}{\rho} + \frac{V_2^2}{2}$$

$$\Rightarrow P_1 + \frac{1}{2}\rho U_x^2 = P_2 + \frac{\rho}{2}(-2U_x \sin\theta)^2$$

$$\Rightarrow P_1 + \frac{1}{2}\rho U^2 = P_2 + \frac{\rho 4 U^2 \sin^2\theta}{2}$$

$$\Rightarrow P_1 - P_2 = 2\rho U^2 \sin^2\theta - \frac{1}{2}\rho U^2$$

$$= U^2 \rho\left(2\sin^2\theta - \frac{1}{2}\right)$$

$$= U^2 \rho\left(\cancel{2}\frac{(1-\cos 2\theta)}{\cancel{2}} - \frac{1}{2}\right)$$

$$P = P_1 - P_2 = U^2 \rho\left(\cos 2\theta - \frac{1}{2}\right)$$

a)
$$F_H = \int_0^{2\pi} p R\, d\theta \cos\theta$$
$$F_V = \int_0^{2\pi} p R\, d\theta \sin\theta$$

$$F_H = \int_0^{2\pi} U^2 \rho \left(\cos 2\theta - \tfrac{1}{2}\right) R \cos\theta\, d\theta$$

$$= U^2 \rho R \left( \underline{\int_0^{2\pi} \cos 2\theta \cos\theta\, d\theta} - \frac{1}{2}\int_0^{2\pi} \cos\theta\, d\theta \right)$$

$$\int_0^{2\pi} \cos 2\theta \cos\theta\, d\theta = \cos\theta \int_0^{2\pi} \cos 2\theta\, d\theta - \int_0^{2\pi} \sin\theta \frac{\sin 2\theta}{2}\, d\theta$$
$$= \cos\theta \int_0^{2\pi} \cos 2\theta\, d\theta - \left[\sin\theta \int_0^{2\pi} \sin 2\theta\, d\theta - \int_0^{2\pi} \frac{\cos 2\theta \cos\theta}{2}\, d\theta\right.$$

Let $\int_0^{2\pi} \cos 2\theta \cos\theta = A$

$$= \cos\theta \int_0^{2\pi} \cos 2\theta\, d\theta - \left[\sin\theta \int_0^{2\pi} \sin 2\theta\, d\theta - \frac{A}{2}\right.$$

$$\frac{A}{2} = \cos\theta \int_0^{2\pi} \cos 2\theta\, d\theta - \sin\theta \int_0^{2\pi} \sin 2\theta\, d\theta$$

$$\frac{A}{2} = \left[\cancel{\cos\theta \frac{\sin 2\theta}{2}}\right]_0^{2\pi} - \left[\cancel{\sin\theta \frac{\cos 2\theta}{2}}\right]_0^{2\pi}$$

$$A = 0$$

$$F_H = U^2 \rho R \left[0 - \frac{1}{2}\cancel{\left[\sin\theta\right]_0^{2\pi}}\right.$$

$$F_H = 0$$

$$F_V = \int_0^{2\pi} p R \sin\theta\, d\theta = \int_{-\pi}^{\pi} U^2 \rho R \left(\cos 2\theta - \tfrac{1}{2}\right) \sin\theta\, d\theta$$

$$= U^2 \rho R \left[\int_{-\pi}^{0} -\left(\cos(-2\theta) - \tfrac{1}{2}\right) \sin(-\theta)\, d\theta + \int_0^{\pi} \left(\cos 2\theta - \tfrac{1}{2}\right) \sin\theta\, d\theta\right]$$

$$= U^2 \rho R \left[\int_0^{\pi} \left(\cos 2\theta - \tfrac{1}{2}\right)(-\sin\theta) + \cancel{\int_0^{\pi} \left(\cos 2\theta - \tfrac{1}{2}\right) \sin\theta\, d\theta}\right]$$

$$F_V = 0$$

$\therefore$ Net force $F_0 = \sqrt{F_H^2 + F_V^2} = 0$

The result makes sense according to De Alembert's Paradox which states that drag force $F = 0$ on a body moving with constant velocity relative to fluid