ADVANCED FLUID MECHANICS
Report on Homework 3
Dimensional analysis, Compressible Flow and Navier-Stokes equations

submitted by
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Computational Mechanics

1. Pumping Crude Oil

   (a) Here we model the pressure gradient in the flow as:

$$\frac{\Delta P}{L} = f(\rho, \vec{v_0}, R, R1, \mu_1, \mu_2, \sigma)$$

Using the Table 1, three quantities with independent dimensions are identified. These quantities, marked in colour, will be used as new basis for forming dimensionless groups with other quantities.

Table 1: Dimensions of different quantities

	$^{\Delta P}/_{L}$	ρ	$\bar{v_0}$	R	R_1	μ_1	μ_2	σ
M	1	1	0	0	0	1	1	1
L	-2	-3	1	1	1	-1	-1	0
T	-2	0	-1	0	0	-1	-1	-2

Based on the above identified basis quantities, the following Π groups can be formed:

$$\Pi_1 = \frac{^{\Delta P}/_{L}\ R_1}{\rho \bar{v_0}^2} = \frac{\Delta P\ R}{L \rho \bar{v_0}^2}$$
$$\Pi_2 = \frac{R}{R_1}$$
$$\Pi_3 = \frac{\rho \bar{v_0} R_1}{\mu_1}$$
$$\Pi_4 = \frac{\rho \bar{v_0} R_1}{\mu_2}$$
$$\Pi_5 = \frac{\sigma}{\rho \bar{v_0}^2 R_1}$$

   (b) Π_3 and Π_4 are correspond to Reynold's number for the given fluids. The fifth Π group can be thought of as ratio of pressure created by surface tension to that of that of momentum density.

$$\Pi_5 = \frac{\sigma}{\rho \bar{v_0}^2 R_1} = \frac{^{\sigma}/_{R_1}}{\rho \bar{v_0}^2}$$

$^{\sigma}/_{R_1}$ term is similar to that of the pressure difference across a bubble film, although, the radius R_1 may not equivalent to that of the curvature of wave.

If we consider the analogy between the interfacial waves and waves in case of a simple ideal string. Here, the role of tension, i.e. the driving force, is played by surface tension. The role of inertia is played by the ρv_0^2 term, which is a measure of kinetic energy. In order to avoid waves, surface tension must overcome inertia forces. Hence, the inequality condition is:

$$\begin{aligned} &\Pi_5 >> 1 \\ \implies &\frac{\sigma}{R_1} >> \rho v_0^2 \end{aligned}$$

(c) Gravity can be neglected in this problem only if the densities are exactly equal, as gravity would have equal pull over both the liquids. Otherwise, gravity could not be neglected, and we would have another Π group :

$$\Pi_6 = \frac{v^2}{gR_1} = \frac{v_0^2/R_1}{g} \text{ or } \frac{v_0}{\sqrt{R_1 g}} = \frac{\rho v_0^2}{\rho R_1 g}$$

Here ρ can be one of the two densities. The sixth non-dimensional group can be expressed in two ways. In the first one, the numerator v_0^2/R_1^2 resembles the expression for centrifugal force. However, in this problem, the flow is strictly axial. As a result, no form of centrifugal forces develop during the flow. However, in the second form, the denominator can be seen as a gravity pressure head. Effectively, we can see it as a ratio of momentum density and hydrostatic pressure head. Here, the gravity forces can be seen as the driving/restoring forces of waves, and the momentum density can be seen as inertia. The hypothesis of neglecting gravity can be true under following condtions:

$\frac{v_0}{\sqrt{R_1 g}} >> 1 \implies$ density is so high that the fluid almost behaves as a bulk motion.

$\frac{v_0}{\sqrt{R_1 g}} << 1 \implies$ restoring force (gravity) is so high that the interface is primarily defined by difference in density & the gravity and no wave behaviour is observed. However, if gravity dominates the flow as well as inter-facial tension, it may be wrong to assume that the two immiscible liquids are separated concentrically and heavier fluid may settle on lower surface of the pipe.

(d) The velocity field is axi-symmetric and identical along z-axis, because of which velocity is just a function of r. Hence, the Navier stokes equations for the steady velocity field $\vec{v} = (0, 0, v_z)$, where $v_z = v_z(r)$ are:

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial r} \\ 0 &= -\frac{1}{r}\frac{\partial p}{\partial \theta} \\ 0 &= -\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \end{aligned}$$

where,

$$\mu = \begin{cases} \mu_1 & \text{if } 0 < r < R_1 \\ \mu_2 & \text{if } R_1 < r < R \end{cases}$$

Hence, $p = p(z)$. The appropriate boundary conditions would be the no-slip condition on the wall and the balance of shear stresses at the interface.

$$v_z\big|_{r=R} = 0$$

and,

$$\tau_{rz}\Big|_{r=R_1^-} = \tau_{rz}\Big|_{r=R_1^+}$$

$$\implies \mu_1 \frac{\partial v_z}{\partial r}\bigg|_{r=R_1^-} = \mu_2 \frac{\partial v_z}{\partial r}\bigg|_{r=R_1^+}$$

The implied condition here is that the velocity v_z is continuous across the interface, although its derivative is discontinuous unless the viscosities are equal.

$$v_z\big|_{r=R_1^-} = v_z\big|_{r=R_1^+}$$

Another condition that arises due to the geometry of the problem is the radial gradient of velocity at the axis of the pipe is zero.

$$\frac{\partial v_z}{\partial r}\bigg|_{r=0} = 0$$

(e) Consider the z component of the Navier Stokes equation, we know that the solutions are of the form-

$$v_z = \begin{cases} \dfrac{1}{4\mu_1}\dfrac{dp}{dz}r^2 + a_1 ln(r) + b_1 & \text{if } 0 < r < R_1 \\ \\ \dfrac{1}{4\mu_2}\dfrac{dp}{dz}r^2 + a_2 ln(r) + b_2 & \text{if } R_1 < r < R \end{cases}$$

Using boundary conditions to obtain constants:

$$\frac{\partial v_z}{\partial r}\bigg|_{r=0} = 0 \implies a_1 = 0$$

$$\mu_1 \frac{\partial v_z}{\partial r}\bigg|_{r=R_1^-} = \mu_2 \frac{\partial v_z}{\partial r}\bigg|_{r=R_1^+} \implies a_2 = 0$$

$$v_z\big|_{r=R} = 0 \implies b_2 = -\frac{1}{4\mu_2}\frac{dp}{dz}R^2$$

$$v_z\big|_{r=R_1^-} = v_z\big|_{r=R_1^+} \implies b_1 = -\frac{1}{4\mu_1}\frac{dp}{dz}(R_1^2 - r^2) - \frac{1}{4\mu_2}\frac{dp}{dz}(R^2 - R_1^2)$$

Hence,

$$v_z = \begin{cases} \dfrac{1}{4\mu_1}\dfrac{\Delta p}{L}(R_1^2 - r^2) + \dfrac{1}{4\mu_2}\dfrac{\Delta p}{L}(R^2 - R_1^2) & \text{if } 0 < r < R_1 \\ \\ \dfrac{1}{4\mu_2}\dfrac{\Delta p}{L}(R^2 - r^2) & \text{if } R_1 < r < R \end{cases}$$

And the interfacial velocity is found to be:

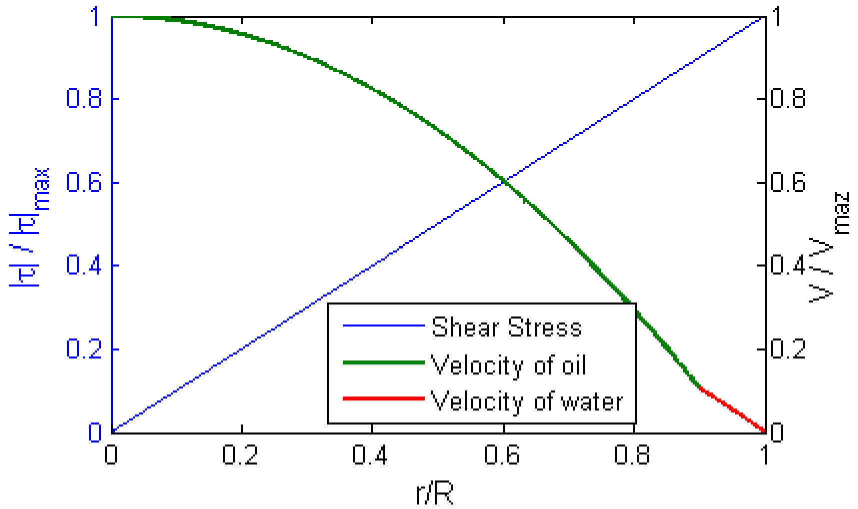
$$v_z\big|_{interface} = \frac{1}{4\mu_2}\frac{\Delta p}{L}(R^2 - R_1^2)$$

The shear stress profile would be:

$$\tau_{rz} = \mu \frac{\partial v_z}{\partial r} = -\frac{1}{2}\frac{\Delta p}{L} r$$

Note that the shear stress is continuous as well as differentiable across the interface.

Figure 1: Velocity and Shear Stress Profiles



(f) The volume flow rate of oil and water can be determined as follows:

$$Q_0 = \int_{A_1} v_z dA = \int_0^{R_1} \left[\frac{1}{4\mu_1}\frac{\Delta p}{L}(R_1^2 - r^2) + \frac{1}{4\mu_2}\frac{\Delta p}{L}(R^2 - R_1^2) \right] (2\pi r) dr$$

$$= \frac{1}{6\mu_2}\frac{\Delta p}{L} R_1^3 + \frac{1}{4\mu_2}\frac{\Delta p}{L}(R^2 - R_1^2) R_1$$

and

$$Q_w = \int_{A_1} v_z dA = \int_{R_1}^{R} \frac{1}{4\mu_2}\frac{\Delta p}{L}(R^2 - r^2)\,(2\pi r) dr$$

$$= \frac{1}{4\mu_2}\frac{\Delta p}{L}\left[\frac{2R^3}{3} - R^2 R_1 + \frac{R_1^3}{3} \right]$$

2. Shock Tube

As soon as the membrane bursts, two pressure fronts travel on either side of membrane equating the pressure and velocity across $x = 0$;, as shown in the Fig 2. As per the small perturbation theory, these pressure fronts travel at the speed c, where $c^2 = \gamma p/\rho$

The $x - t$ diagram can be drawn using the fact that the speed of propagation is c, hence, the velocity in the region $|x| > ct$ is unaffected. For $|x| < xt$, the fluid moves with a constant velocity determined by the strength of the shocks i.e. the pressure difference.
After time $t_r = l/c$ the pressure front on the right side reflects off the wall at $x = l$. The pressure wave travels backwards, the position of which can be expressed as $x = 2l - ct$. Hence, $x - t$ looks as in ??. Velocity in region 1 is doesn't change because no pressure front has crossed this region. Region 2 has a finite velocity which can be calculated using the momentum equation.

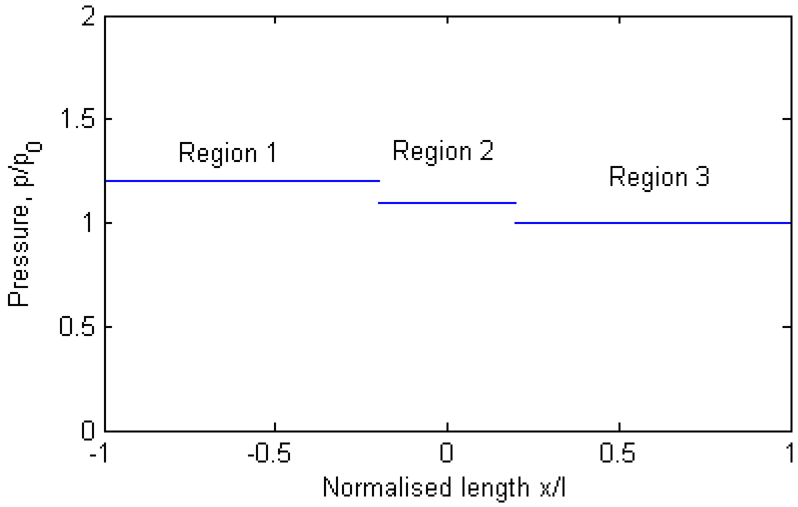


Figure 2: Pressure Front after certain time t

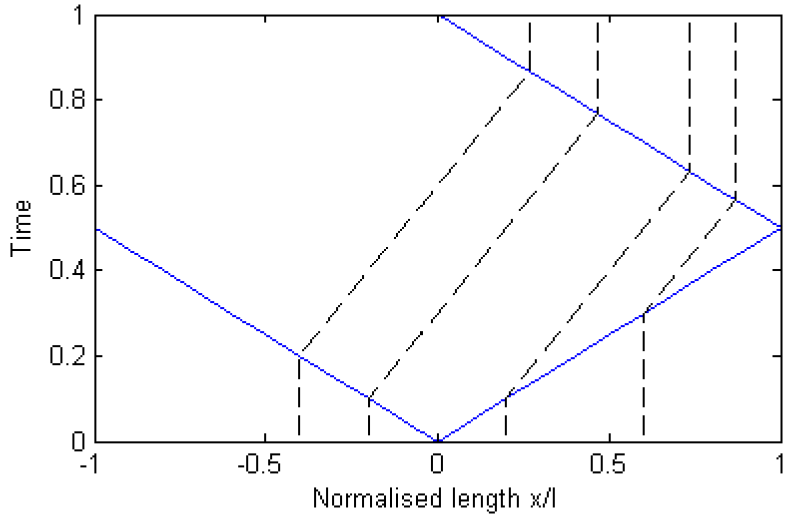


Figure 3: x-t Diagram

Here we assume, $\rho_1 \approx \rho_0$ and $\rho_2 \approx \rho_0$, because $(p_1 - p_0)/p_0 << 1$.

We use Rankine Hugnoit relation $p + \rho u^2 = constant$ across the each wave. In order to use this relation, we need to use the velocity relative to the shock. Hence, for the left running shock we have,

$$
\begin{aligned}
& p_1 + \rho_0 c^2 = p_2 + \rho_0 (c+u)^2 \\
\implies & \frac{p_1 - p_2}{\rho} = (c+u)^2 - c^2 \\
\implies & \left(\frac{p_1 - p_2}{p_0}\right) \frac{p_0}{\rho} = u^2 + 2uc \\
\implies & \frac{1}{\gamma}\left(\frac{p_1 - p_2}{p_0}\right) = \frac{u}{c}\left(1 + \frac{2u}{c}\right) \approx \frac{u}{c}
\end{aligned}
$$

Similarly, for the right running pressure front, we have:

$$\frac{1}{\gamma}\left(\frac{p_2 - p_0}{p_0}\right) = \frac{u}{c}\left(1 - \frac{2u}{c}\right) \approx \frac{u}{c}$$

Combining both the equations, we have:

$$\frac{u}{c} = \frac{1}{2\gamma}\left(\frac{p_1 - p_0}{p_0}\right)$$

and

$$p_2 = \frac{p_0 + p_1}{2}$$

After the right running wave reaches $x = l$ at $t = l/c$, it is reflected and then travels leftward with the same speed c. Region 3 of the figure ?? is the region after the wave is reflected. We use the boundary condition, $u = 0$ at $x = l$. Since, the velocity behind the wave must be uniform, we can say $u = 0$ in Region 3.
Using this information, we again use the Rankine-Hugnoit relation across this wave to evaluate the pressure.

$$\begin{aligned}
& p_3 + \rho\left(c + \cancelto{0}{u_3}\right)^2 = p_2 + \rho(c - u)^2 \\
\Longrightarrow & \left(\frac{p_3 - p_2}{p_0}\right)\frac{p_0}{\rho}\frac{1}{c^2} = \frac{u}{c}\left(1 - \frac{2u}{c}\right) \\
\Longrightarrow & \frac{1}{\gamma}\left(\frac{p_3 - p_2}{p_0}\right) = \frac{u}{c}\left(1 - \frac{2u}{c}\right) \approx \frac{u}{c} \\
\Longrightarrow & \frac{1}{\gamma}\left(\frac{p_3 - p_2}{p_0}\right) = \frac{1}{2\gamma}\left(\frac{p_1 - p_0}{p_0}\right) \\
\Longrightarrow & p_3 = p_0
\end{aligned}$$

Conclusion: In the region behind the reflected wave, we have $v = 0$ and $p = p_1$.