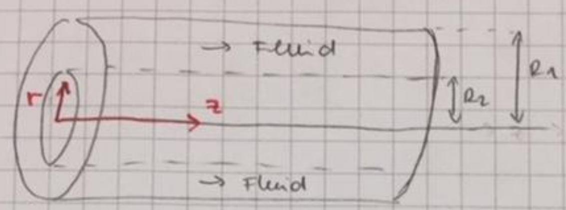
AFM

Homework 2

1)



→ Renaming of coordinates!
assumptions:
- steady - laminar - incompressible flow
- pressure-driven
- flow along z-axis
→ using cylindrical coord. $\begin{pmatrix} r \\ \theta \\ z \end{pmatrix}$

a) $\vec{V} = [\underbrace{V_r(r,\theta,z)}_{=0}, \underbrace{V_\theta(r,\theta,z)}_{=0, \text{ along } z\text{-axis}}, V_z(r,\theta,z)] \rightarrow V_z(r,z)$ irrotational flow

mass conservation:
(for incompressible flow)

$$\nabla \cdot \vec{V} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\underbrace{\cancel{(rV_r)}}_{V_r=0} + \underbrace{\cancel{\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}}}_{V_\theta=0} + \frac{\partial V_z}{\partial z} = 0$$

$$① \rightarrow \frac{\partial V_z}{\partial z} = 0$$

→ V_z independent of z, $\boxed{V_z(r)}$

Navier-Stokes equation (only in z-direction as other 2 direction do not give any additional information → $V_r = V_\theta = 0$):

$$z: \rho\left(\underbrace{\cancel{\frac{\partial V_z}{\partial t}}}_{\text{steady}}^{=0} + \underbrace{\cancel{V_r\frac{\partial V_z}{\partial r}}}_{V_r=0}^{=0} + \underbrace{\cancel{\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}}}_{V_\theta=0}^{=0} + \underbrace{V_z\cancel{\frac{\partial V_z}{\partial z}}}_{\text{mass cons.}}^{=0}\right) = -\frac{\partial p}{\partial z} + \mu \nabla^2 V_z + \underbrace{\cancel{\rho b_z}}_{\text{no body forces in } z\text{-dir.}}^{=0}$$

$$\frac{\partial p}{\partial z} = \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}V_z\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(V_z)}}_{V_z(r)}^{=0} + \underbrace{\cancel{\frac{\partial^2}{\partial z^2}V_z}}_{V_z(r)}^{=0}\right)$$

$$② \rightarrow \frac{\partial p}{\partial z} = \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}V_z\right)\right)$$

① and ② are the equations governing the flow

b) B.C: no-slip-condition at the walls: $V_z = 0$ at $r = R_2$

$V_z = 0$ at $r = R_1$

c) ②: $\frac{r}{\mu}\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}V_z\right) \quad | \int dr$

$$B_1 + \frac{r^2}{2\mu}\frac{\partial p}{\partial z} = r\frac{\partial}{\partial r}V_z$$

$$\frac{B_1}{r} + \frac{r}{2\mu}\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}V_z \quad | \int dr \Rightarrow V_z(r) = \frac{\partial p}{\partial z}\frac{r^2}{4\mu} + B_1 \ln(r) + B_2$$

inserting BC to get B_1, B_2:

$V(r=R_1)=0$ $\qquad\qquad$ $V(r=R_2)=0$

$$0=\frac{\partial p}{\partial z}\frac{R_1^2}{4\mu}+B_1\ln(R_1)+B_2 \qquad 0=\frac{\partial p}{\partial z}\frac{R_2^2}{4\mu}+B_1\ln(R_2)+B_2$$

$$B_2=-\frac{\partial p}{\partial z}\frac{R_1^2}{4\mu}-B_1\ln(R_1) \longrightarrow B_1=\frac{\partial p}{\partial z}\frac{(R_1^2-R_2^2)}{4\mu\ln(R_2/R_1)}$$

$$B_2=-\frac{\partial p}{\partial z}\frac{1}{4\mu}\left(\frac{\ln(R_1)(R_1^2-R_2^2)}{\ln(R_2/R_1)}+R_1^2\right)$$

$$\Rightarrow V_z(r)=\frac{\partial p}{\partial z}\frac{1}{4\mu}\left(r^2-R_1^2+\frac{(R_1^2-R_2^2)}{\ln(R_2/R_1)}\cdot\ln\left(\frac{r}{R_1}\right)\right)$$

radius of maximum velocity:

$$\frac{dV_z}{dr}=\frac{\partial p}{\partial z}\frac{r}{2\mu}+\frac{B_1}{r}=\frac{\partial p}{\partial z}\frac{1}{2\mu}\left(r+\frac{(R_1^2-R_2^2)}{2r\ln(R_2/R_1)}\right)\overset{!}{=}0$$

$$r^*=\pm\sqrt{\frac{(R_2^2-R_1^2)}{2\ln(R_2/R_1)}}$$

maximum $\Leftrightarrow V''_z(r^*)<0$

$\rightarrow$ 2 radii of max. velocity as flow is symmetric

$$\Leftrightarrow \frac{\partial p}{\partial z}<0$$

d) volume flow rate: $Q=A\cdot V=\int_0^{2\pi}\int_{R_2}^{R_1} V_z(r)\; r\,dr\,d\theta$

$$=2\pi\int_{R_2}^{R_1}\frac{\partial p}{\partial z}\frac{r^3}{4\mu}+B_1\underbrace{\ln(r)\cdot r}_{(*)}+B_2\cdot r\;dr$$

$$(*):\int_{R_2}^{R_1}\ln(r)\cdot r\,dr=\left[r(r\ln(r)-r)\right]_{R_2}^{R_1}-\int_{R_2}^{R_1} r\ln(r)-r\,dr$$

$$2\cdot\int_{R_2}^{R_1}\ln(r)\cdot r\,dr=\left[r^2(\ln(r)-1)\right]_{R_2}^{R_1}-\int_{R_2}^{R_1} r\,dr$$

$$\int_{R_2}^{R_1}\ln(r)\cdot r\,dr=\frac{1}{2}\left[r^2(\ln(r)-1)\right]_{R_2}^{R_1}-\frac{1}{4}\left[r^2\right]_{R_2}^{R_1}$$

$$\underline{Q}=\pi\cdot\frac{\partial p}{\partial z}\frac{1}{2\mu}\cdot\left[\frac{R_1^4}{4}-\frac{R_2^4}{4}-\left(\frac{\ln(R_1)(R_1^2-R_2^2)}{\ln(R_2/R_1)}+R_1^2\right)\frac{(R_1^2-R_2^2)}{2}\right.$$

$$\left.+\frac{(R_1^2-R_2^2)}{\ln(R_2/R_1)}\left(\frac{R_1^2}{2}\left(\ln(R_1)-\frac{1}{2}\right)-\frac{R_2^2}{2}\left(\ln(R_2)-\frac{1}{2}\right)\right)\right]$$

$$=\frac{\partial p}{\partial z}\frac{\pi}{2\mu}\left[\frac{R_1^4}{4}-\frac{\phi^4R_1^4}{4}+\frac{\phi^2R_1^2\ln(R_1)-R_1^2\ln(\phi R_1)}{2\ln(\phi)}(R_1^2-\phi^2R_1^2)\right.$$

$\phi=\frac{R_2}{R_1}$

$R_2=\phi R_1$

$$\left.+\frac{(R_1^2-\phi^2R_1^2)}{2\ln(\phi)}\left(R_1^2\left(\ln(R_1)-\frac{1}{2}\right)-\phi^2R_1^2\left(\ln(\phi R_1)-\frac{1}{2}\right)\right)\right]$$

e)

$$\lim_{\phi\to 0} \frac{\partial p}{\partial z}\frac{\pi}{2\mu}\Bigg[\frac{R_1^4}{4} - \overbrace{\frac{\phi^4 R_1^4}{4}}^{\nearrow 0} + \frac{\phi^2 R_1^2 \ln(R_1) - R_1^2 \ln(\phi R_1)}{2\ln(\phi)}(R_1^2-\phi^2 R_1^2)$$

$$\overset{0}{+}\; \underset{\to -\infty}{\frac{(R_1^2-\phi^2 R_1^2)}{2\ln(\phi)}} \left(R_1^2 \Big(\underbrace{\ln(R_1)}_{\to 0} \underset{\to 0}{-\tfrac{1}{2}}\Big) - \phi^2 R_1^2\Big(\underbrace{\ln(\phi R_1)}_{\to 0} - \underset{0}{\tfrac{1}{2}}\Big)\right)\Bigg]$$

single terms:

$$\lim_{\phi\to 0}\frac{\overset{\to 0}{\phi^2 R_1^4 \ln(R_1)}}{\underset{\to -\infty}{2\ln(\phi)}} \overset{\text{del'Hop.}}{=} \lim_{\phi\to 0}\frac{2\phi R_1^4 \ln(R_1)}{2\,\frac{1}{\phi}} = 0 \to \text{same for terms multiplying } (-\phi^2 R_1^2)$$

$$\lim_{\phi\to 0}\frac{-R_1^4 \ln(\phi R_1)}{2\ln(\phi)} = \lim \frac{-R_1^4 \cancel{\ln(\phi)}}{2\cancel{\ln(\phi)}} - \underbrace{\frac{R_1^4 \ln(R_1)}{2\ln(\phi)}}_{\to 0}{}_{\to -\infty}$$

$$= -\frac{R_1^4}{2}$$

↳ using de l'Hopital as previously, one can see that all the other terms from above tend to 0 as $\phi \to 0$

all in all:

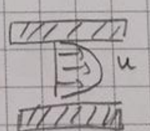
$$\lim_{\phi\to 0}(Q) = \frac{\partial p}{\partial z}\frac{\pi}{2\mu}\left[\frac{R_1^4}{4} - \frac{R_1^4}{2}\right] = -\frac{\partial p}{\partial z}\frac{\pi}{2\mu}\frac{R_1^4}{4}$$

for $R_1 = \frac{D}{2}$: $\underline{\lim_{\phi\to 0}(Q) = -\frac{\partial p}{\partial z}\frac{\pi D^4}{128\mu}}$

↳ this is the formula of volume flow rate for circular Poiseuille flow.

$\phi \to 0 \iff R_1 \gg R_2$ → in this case we have a flow in a cylindrical tube instead of a flow in the annular space of two coaxial tubes

↳ usual Poiseuille case.



2) boundary layer $\delta(x)$

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2$$ with $u = 0$ at $y = 0$

$u = U$, $\frac{\partial u}{\partial y} = 0$ at $y = \delta$

$\underline{a = 0}$

$1 = b + c \qquad \frac{\delta u}{\delta y} = \frac{Ub}{\delta} + \frac{2ycU}{\delta^2} \rightarrow -2c = b \rightarrow b = -2c$

$1 = -2c + c \rightarrow \underline{c = -1}, \quad \underline{b = 2}$

$$\rightarrow \frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

Kármán-Polhausen: drag force on a flate plate with momentum thickness θ:

$D(x) = \rho b U^2 \theta$ for uniform flow

b... thickness plate

and $\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$

$$\rightarrow \frac{dD(x)}{dx} = \rho b U^2 \frac{d\theta}{dx}$$

drag force $\hat{=}$ integrated shear stress along the plate:

$$D(x) = b\int_0^x \tau_w(x)\, dx \rightarrow \frac{dD(x)}{dx} = b\tau_w(x)$$

momentum-integral relation
for flat plate boundary layer: $\tau_w(x) = \rho U^2 \frac{d\theta}{dx}$

here: $\theta = \int_0^\delta \left(2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)\left(1 - 2\frac{y}{\delta} + \left(\frac{y}{\delta}\right)^2\right) dy$

$$= \int_0^\delta \left(2\frac{y}{\delta} - 5\frac{y^2}{\delta^2} + 4\frac{y^3}{\delta^3} - \frac{y^4}{\delta^4}\right) dy$$

$$= \left[\frac{y^2}{\delta} - \frac{5}{3}\frac{y^3}{\delta^2} + \frac{y^4}{\delta^3} - \frac{1}{5}\frac{y^5}{\delta^4}\right]_0^\delta = \frac{2}{15}\cdot\delta \rightarrow \frac{d\theta}{dx} = \frac{2}{15}\frac{d\delta}{dx}$$

$$\tau_w = \mu\frac{\partial u}{\partial y}\Big|_{y=0} = \mu\left(\frac{2U}{\delta} - \frac{2Uy}{\delta^2}\right)\Big|_{y=0} = \frac{2U\mu}{\delta}$$

$$\rightarrow \tau_w = \rho U^2 \frac{d\theta}{dx}$$

$$\frac{2U\mu}{\delta} = \rho U^2 \frac{2}{15}\frac{d\delta}{dx}$$

$$\int_0^x \frac{15}{U}\frac{\mu}{\rho}\, dx = \int_0^\delta \delta\, d\delta$$ with $\delta(x) = 0$ at $x = 0$

$$\frac{15}{U}\nu x = \frac{1}{2}\delta^2$$

$$\delta = \sqrt{\frac{30\nu x}{U}} \quad \Big|\cdot\frac{1}{x}$$

$$\frac{\delta}{x} = 5{,}477\cdot Re_x^{-\frac{1}{2}} \approx \frac{5.0}{Re_x^{1/2}}$$ Blasius

→ our solution is a good approximation to Blasius' exact solution.