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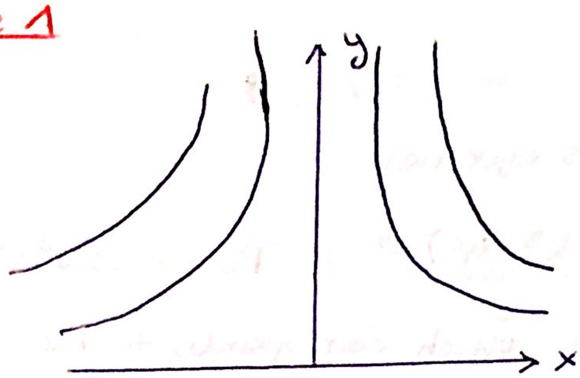
AFM : Homework 4: Navier Stokes and Bound. Layer

Exercise 1



a) ideal fluid

$$\psi(r,\theta) = Ur^2 \sin(2\theta)$$

- compute the velocity field in cartesian coordinates (u, v)
- show verifies BC
- expression for pressure distribution

Velocity in polar

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \rightarrow V_r = 2Ur\cos(2\theta)$$

$$V_\theta = -\frac{\partial \psi}{\partial r} \longrightarrow V_\theta = -2Ur\sin(2\theta)$$

$$x = r\cos(2\theta)$$
$$y = r\sin(2\theta)$$

$$\left.\begin{aligned} u &= 2Ux \\ y &= -2Uy \end{aligned}\right\}$$ velocity field in cartesian coordinates

The boundary may be considered to be curved, such as the surface of a circular cylinder, provided the region under consideration is small in extent compared with the radius of curvature of the surface.

Bernouilli's equations

$$\frac{P}{\rho} + \frac{V_2^2}{2} + \cancel{gz_2} = \frac{P_0}{\cancel{\rho}} + \frac{\cancel{V_1}^{2}}{2}^{=0} + \cancel{gz_1}$$ when impinges

$$P = \left(\frac{P_0}{\rho} - \frac{V_2^2}{2}\right)\cdot\rho = P_0 - \frac{V_2^2\rho}{2} =$$

$V = \sqrt{u^2 + v^2}$ $\quad V = \sqrt{(2Ux)^2 + (-2Uy)^2} = \sqrt{4U^2x^2 + 4U^2y^2} =$

$V = 2U\sqrt{x^2+y^2}$

if we replace it in the Bernouilli's equation:

$$P = Po - \frac{V^2}{2}\rho = Po - \frac{4U^2(x^2+y^2)\rho}{2} = Po - 2U^2(x^2+y^2)\rho$$

where po is the Bernouilli constant which corresponds to the pressure at the stagnation point.

The velocity and pressure distributions satisfy the potential-flow problem, and also the equations of motion for a viscous, incompressible fluid.

b) But for potential flows $u = \nabla \phi$, so that:

$$\nabla^2 u = \nabla^2 (\nabla \phi) = \nabla (\nabla^2 \phi) = 0$$

That is the viscous-shear terms of the Navier-Stokes eq.

But potential-flow don't satisfy the no-slip boundary condition.

c) So we have to modify the potential-flow field in such a way that meeting this boundary condition would be possible.

The x component of velocity is taken to be $u = 2Ux\,f'(y)$

$f'(y) = \frac{\partial f}{\partial y}$, then the continuity equation requires that

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -2Uf'(y).$$

so that the vertical component of the velocity will be of the form:

$V = -2U.(y)$

Defining the velocity field in this way satisfies the continuity equation for all functions $f(y)$ and if we stipulate that $f(y) \to y$ as $y \to \infty$, the potential-flow solution will be recovered far from the boundary.

d)
The equations of motion have to be satisfied, this will impose restrictions on the function f. The equations to be satisfied are:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

substituting:

$$4U^2x(f')^2 - 4U^2x f f'' = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2Uxf'''$$

$$4U^2 f f' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2U\nu f''$$

The second of these eq's will be used to establish the pressure distribution. Integrating the last equation with respect to y gives the following expression for the pressure:

$$p(x,y) = -2\rho U^2 (f)^2 - 2\rho U\nu f' + g(x)$$

$g(x)$ is some function of x which may be determined by comparison with the potential-flow pressure distribution which should be recovered for large values of y.

$f(y) \to y$ for large values of y shows that, for large values of y

$$p(x,y) \longrightarrow -2\rho U^2 y^2 - 2\rho U\nu + g(x)$$

for comparison with the potential-flow pressure, requires that:

$$g(x) = p_0 - 2\rho U^2 x^2 + 2\rho U\nu$$

<u>Pressure distrib.</u> in viscous fluid will be:

$$p(x,y) = p_0 - 2\rho U^2 (f)^2 + 2\rho U\nu(1 - f') - 2\rho U^2 x^2$$

e) we have satisfied the continuity eq and the eq. of the y momentum.

$\frac{\partial p}{\partial x} = -4\rho U^2 x$, so that he equation of x momentum becomes:

$$4U^2x(f')^2 - 4U^2x f f'' = 4U^2x + 2U\nu x f'''$$

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this becomes:

$$\frac{\nu}{2U} f''' + f f'' - (f')^2 + 1 = 0$$

The BC $u(x,0) = 0$ requires that $f'(0) = 0$ and the condition $v(x,0) = 0$ requires $f(0) = 0$.

Also the condition, that the potential-flow solution be recovered as $y \to \infty$ requires that $f(y) \to y$ or that $f'(y) \to 1$ as $y \to \infty$.

$$f(0) = f'(0) = 0$$

$$f'(y) \to 1 \text{ as } y \to \infty$$

making the following change of variables:

$$\phi(\eta) = \sqrt{\frac{2U}{\nu}}\, f(\eta)$$

$$\eta = \sqrt{\frac{2U}{\nu}}\, y$$

In terms of $\phi(\eta)$ the problem to be solved is:

$$\left.\begin{aligned} &\phi''' + \phi\phi'' - (\phi')^2 + 1 = 0 \\ &\phi(0) = \phi'(0) = 0 \\ &\phi'(\eta) \to 1 \text{ as } \eta \to \infty \end{aligned}\right\} \quad (1)$$

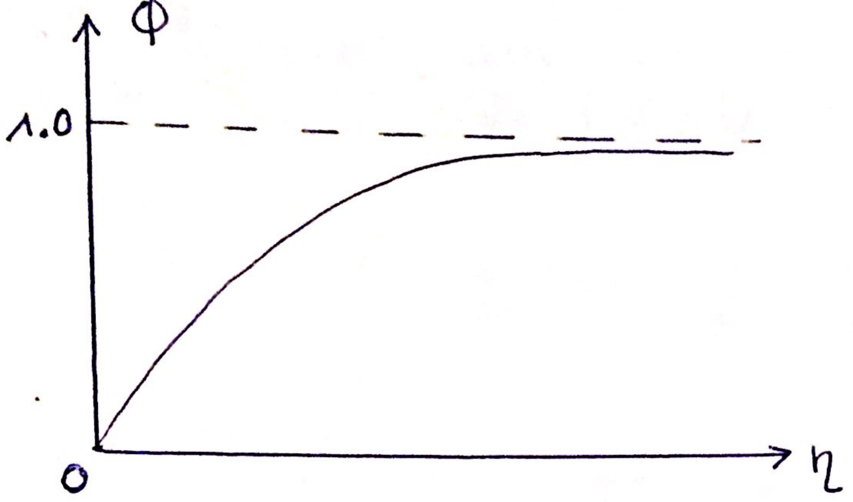
the velocity and pressure fields in stagnation-point flow are given by:

$$u(x,y) = \phi'$$

$$v(x,y) = -\sqrt{2U\nu}\, \phi$$

$$p(x,y) = p_0 - \rho U \nu \phi^2 + 2\rho U \nu (1 - \phi') - 2\rho U^2 x^2$$

$\phi(\eta)$ is the solution of the eqs (1) and $\eta = \sqrt{2U}\, y / \sqrt{\nu}$



if we iterate, the value of η for which $\phi' = 0'99$ is about $2'4$

The potential-flow is recovered when $\eta = 2'4$. Then if δ denotes the value of y at this edge of the viscous layer, it follows that:

$$\sqrt{\frac{2U}{\nu}}\,\delta = 2'4 \qquad \delta = 2'4\sqrt{\frac{\nu}{2U}}$$

Exercise 2

Use Kármán - Pohlhausen approximation to compute the boundary layer solution for an uniform flow over a flat plate. Assume a quadratic polynomial form for the velocity profile:

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2$$

and use the following Boundary conditions:

$$u = 0 \quad \text{at } y = 0$$
$$u = U, \ \frac{\partial u}{\partial y} = 0 \quad \text{at } y = \delta$$

Compare the results with the exact Blasius solution and with the ones obtained assuming a cubic velocity profile.

- quadratic:

$$\frac{u}{U} = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2$$

$u=0$ for $y=0$ $\Rightarrow$ $\frac{u}{U} = 0$ for $\frac{y}{\delta} = 0$

$u = U$ for $y = \delta$, $\frac{\partial u}{\partial y} = 0$ $\Rightarrow$ $\frac{u}{U} = 1$ $\frac{\partial u}{\partial y} = 0$ for $\frac{y}{\delta} = 1$

- $0 = a + b \cdot 0 + c \cdot 0 \Rightarrow a = 0$
- $1 = b(1) + (1)^2 \cdot c$
- $0 = b + 2c(1)$

$1 = -2c + c$ $\qquad b = 2$, $c = -1$

so we have:

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$

Momentum integral equation (flat plate): $\frac{d}{dx}\int_0^\delta u\,(U-u)\,dy = \frac{\tau_0}{\rho}$

with:

$$\tau_0 = \mu \frac{U}{\delta} \left.\frac{\partial(u/U)}{\partial(y/\delta)}\right|_{y/\delta = 0}$$

$$\frac{d}{dx}\int_0^{\delta} u\left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)\left(U-U\left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)\right)dy=\frac{\tau_0}{\rho}$$

$$U^2\frac{d}{dx}\int_0^{\delta}\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right)dy=\frac{\tau_0}{\rho}\ ;\ \tau_0=\mu\frac{U}{\delta}$$

$$\frac{d}{dx}\int_0^{\delta}\left(\frac{2y}{\delta}-\frac{5y^2}{\delta^2}+\frac{4y^3}{\delta^3}-\frac{y^4}{\delta^4}\right)dy=\frac{\nu}{\delta U}$$

$$\frac{d}{dx}\left[\frac{2y^2}{2\delta}-\frac{5y^3}{3\delta^2}+\frac{4y^4}{4\delta^3}-\frac{y^5}{5y^4}\right]_0^{\delta}=\frac{2\nu}{\delta U}\rightarrow$$

$$\rightarrow\frac{d}{dx}\left(\frac{\cancel{2}\delta}{15}\right)=\frac{\cancel{2}\nu}{\delta u}$$

$$\boxed{\frac{d}{dx}(\delta)=\frac{15\nu}{\delta U}}$$ Ordinary dif. Equation

$$\delta\cdot d\delta=\frac{15\nu}{U}dx\rightarrow\int\delta\, d\delta=\int\frac{15\nu}{U}dx\rightarrow\frac{\delta^2}{2}=\frac{15\nu x}{U}+c$$

If $x=0\rightarrow\delta=0\longrightarrow\frac{0^2}{2}=\frac{15\nu(0)}{U}+c\longrightarrow c=0$

$$\delta=5'477\sqrt{\frac{\nu x}{U}}$$

The bondary layer thickness can be expressed in adimentional form as:

$$\frac{\delta}{x}=\frac{5'477}{x}\sqrt{\frac{\nu x}{U}}\longrightarrow\frac{\delta}{x}=\frac{5'477}{\sqrt{Re}}$$

- Kármán - Pohlhausen (cubic)

$$\frac{u}{U}=a+b\left(\frac{y}{\delta}\right)+c\left(\frac{y}{\delta}\right)^2+d\left(\frac{y}{\delta}\right)^3$$

- $u=0$ at $y=0$
- $\frac{\partial^2 u}{\partial y^2}$ at $y=0$
- $u=U$ at $y=\delta$; $\frac{\partial u}{\partial y}=0$ at $y=\delta$

We know that $\frac{\partial^2 u}{\partial y^2}$ at $y=0$ because: from the x-momentum equation particularizated in $y=0$.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \rightarrow \frac{\partial^2 u}{\partial y^2} = 0 \ (y=0)$$

$u(0)=0$

Pressure doesn't depend on "x".

B.C adimensionalitation:

- $\frac{u}{U} = 0$ at $\frac{y}{\delta} = 0 \rightarrow 0 = a + b(0) + c(0) + d(0) \rightarrow a = 0$
- $\frac{\partial^2 (\frac{u}{U})}{\partial (y/\delta)^2}$ at $\frac{y}{\delta} = 0 \rightarrow 0 = 2c + 6d(0) \rightarrow c = 0$
- $\frac{u}{U} = 1$ at $\frac{y}{\delta} = 1 \rightarrow 1 = b(1) + d(1)^3 \rightarrow d = -1/2$
- $\frac{\partial (u/U)}{\partial (y/\delta)} = 0$ at $\frac{y}{\delta} = 1 \rightarrow 0 = b + 3d(1)$ $\quad b = 3/2$

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Momentum integral equation (flate plate)

$$\frac{d}{dx}\int_0^\delta \left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1 - \frac{3}{2}\left(\frac{y}{\delta}\right) + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy = \frac{\tau_0}{\rho U^2}$$

$$\frac{d}{dx}\left[\frac{3}{2}\cdot\frac{y^2}{2\delta} - \frac{9}{4}\frac{y^3}{3\delta^2} + \frac{3}{4}\frac{y^5}{5\delta^4} - \frac{1}{2}\frac{y^4}{4\delta^3} + \frac{3}{4}\cdot\frac{y^5}{5\delta^3} - \frac{1}{4}\frac{y^7}{7\delta^6}\right]_0^\delta$$

$$= \frac{\tau_0}{\rho U^2}$$

$$\frac{d}{dx}\left(\frac{39\delta}{280}\right) = \frac{\tau_0}{\rho U^2}$$

$$\tau_0 = \mu\cdot\frac{U}{\delta}\left(\frac{3}{2}\left(1 - \cancel{\left(\frac{y}{\delta}\right)^2}^{\,0}\right)\right) = \mu\frac{U}{\delta}\cdot\frac{3}{2}$$

$$\frac{d}{dx}\left(\frac{39\delta}{280}\right) = \frac{3\mu}{2\rho U\delta}$$

$$\boxed{\frac{d\delta}{dx} = \frac{140}{13}\cdot\frac{\nu}{U\delta}} \rightarrow \text{ODE}$$

Resolution

$$\frac{d\delta}{dx} = \frac{140}{13}\frac{\nu}{U\delta} \qquad \delta\, d\delta = \frac{140}{13}\cdot\frac{\nu}{U}dx \rightarrow$$

$$\rightarrow \int \delta\, d\delta = \int \frac{140}{13}\frac{\nu}{U}dx \rightarrow \frac{\delta^2}{2} = \frac{140}{13}\frac{\nu}{U}x + \cancel{c}$$

$$\delta^2 = \frac{280}{13}\cdot\frac{\nu}{U}x \rightarrow \delta = 4'641\sqrt{\frac{\nu x}{U}}$$

$\delta(0)=0$

$0^2 = \frac{140}{13}\cdot\frac{\nu}{U}(0)+c$

$c=0$

The boundary layer thickness is expresed in adimensional form as:

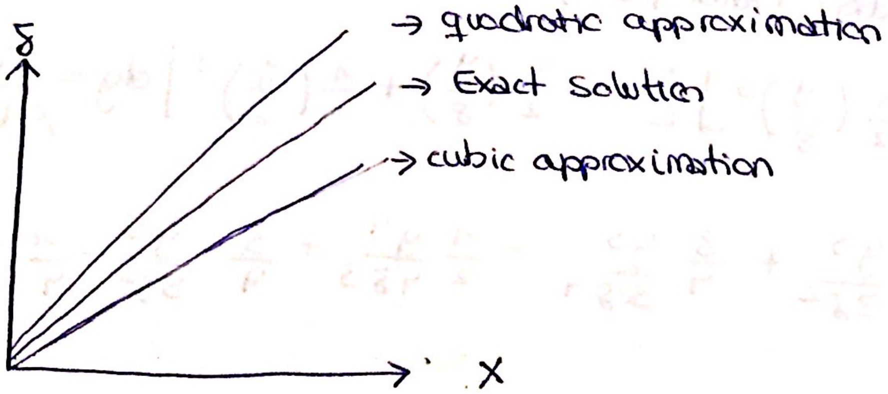
$$\frac{\delta}{x} = \frac{4'641}{x}\sqrt{\frac{\nu x}{U}} \rightarrow \boxed{\frac{\delta}{x} = \frac{4'641}{\sqrt{Re}}}$$

- Blasius exact solution:

$$\frac{\delta}{x} = \frac{5}{\sqrt{Re}}$$

If we compere tho previous approximations to the exact solution we can observe that the cubic approximation is closer, as expected.

we plot tho results:



as we can see the quadratic approximation gives a wider boundary layer with respect to the exact solution. And the cubic gives a thinner one.