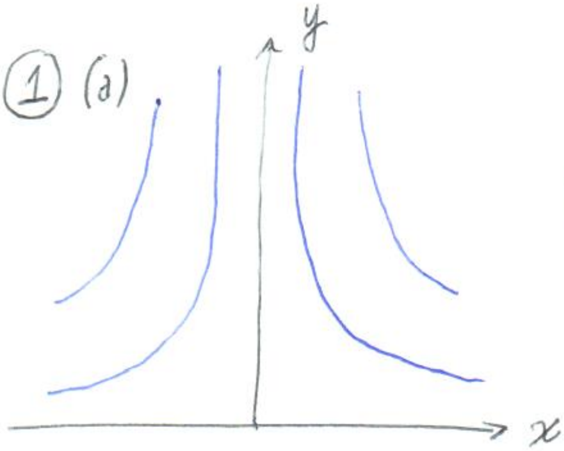
ADVANCED FLUIDS MECHANICS - HOMEWORK 4

ALBERT TAULERA CAMPOS   Due 30th Dec.

(1) (a)



$\psi(r,\theta) = Ur^2 \sin(2\theta)$

The velocity field can be computed easily in cartesian coordinates, and will be like:

$u = 2Ux$
$v = -2Uy$ from $\psi(x,y) = 2Uxy$

(The equivalent in cartesian coordinates ...)

The boundary conditions to be applied are the following:

- $v(y=0) = 0$ (no penetration)
- $u(y=0) = 0$ (no-slip) [viscosity)
- $u = v = 0$ at $y = 0$, $x = 0$ (stagnation point) $\longrightarrow$ This one is not required, as we already impose them using the other ones

We can see that the no penetration condition is fullfilled correctly $[v(y=0) = -2U \cdot 0 = 0]$ but not the no-slip boundary condition.

Pressure distribution

Using the Bernoulli formula between the stagnation point (1) and any other one (2):

$$\int_1^2 \cancel{\frac{\partial v}{\partial t}}^{\,0} ds + \left(\frac{1}{2}v_2^2 + \frac{p_2}{\rho} + \cancel{gz_2}^{\,0}\right) - \left(\cancel{\frac{1}{2}v_1^2}^{\,0} + \frac{p_1}{\rho} + \cancel{gz_1}^{\,0}\right) = 0$$

Reordering:

$$\frac{p_2}{\rho} = \frac{p_1}{\rho} - \frac{1}{2}v_2^2 \Rightarrow p_2 = p_1 - \frac{\rho}{2}v_2^2 \Rightarrow p_2 = p_1 - \frac{\rho}{2}(u^2 + v^2) \Rightarrow$$

$$\Rightarrow p_2 = p_1 - \frac{\rho}{2}\cdot(4U^2x^2 + 4U^2y^2) \Rightarrow p_2 = p_1 - 2\rho U^2(x^2+y^2)$$

b) Navier-Stokes in cartesian coordinates:

Mass conservation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad 2U - 2U + 0 = 0 \quad \checkmark$$

Momentum conservation

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\nabla^2 u + \rho b_x$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\nabla^2 v + \rho b_y$$

$$\rho \cdot (0 + 2Ux \cdot 2U + 0 + 0) = -(-4\rho U^2 x) \quad \checkmark$$

$$\rho\,(0 + 0 + (-2Uy)(2U) + 0) = 2\rho U^2 2y \quad \checkmark$$

Both mass and momentum N-S equations fullfill the requirements.

As shown before, the viscous boundary condition at $y=0$ for the horizontal velocity u is not fullfilled, because we have-slip-condition

¿ $u(y=0) = 0$? $\rightarrow$ $u(y=0) = 2Ux$, so ~~not~~ we would need something else to be able to work with it.

c) $u = 2Ux\,f'(y)$ The appropriate boundary conditions would be:

$v = -2U f(y)$

$u(x,0) \rightarrow f'(0) = 0$ and $v(x,0) = 0 \rightarrow f(0) = 0$

at $y \rightarrow \infty$ $f(y) \rightarrow y$ or $f'(y) \rightarrow 1$ as $y \rightarrow \infty$

Then:

B.C $\quad \boxed{f(0) = f'(0) = 0} \quad$ & $\quad \boxed{f'(y) \rightarrow 1 \text{ as } y \rightarrow \infty}$

$\left(\lim_{y \to \infty} f'(y) = 1\right)$

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d) y-momentum → Pressure in terms of f.

$$\rho\left(\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \cancel{\mu\nabla^2 v} + \cancel{\rho b_y}$$

$$\rho\left(v\frac{dv}{\partial y}\right) = -\frac{\partial p}{\partial y}$$

$$\rho\left((-2Uf(y))(-2Uf'(y)\right) = -\frac{\partial p}{\partial y}$$

$$-\frac{\partial p}{\partial y} = \rho\left(4U^2 f(y)\cdot f'(y)\right)dy \quad \text{Integrating by parts:}$$

$$+p = -\rho 4U^2\frac{f^2(y)}{2} + c_1 = -\rho 2U^2 f^2(y) + c_1$$

Applying the condition:

$$\lim_{y\to\infty} -2\rho U^2 \overset{\infty}{f^2(y)} + c_1 = p_1 - 2\rho U^2(x^2 + \overset{\infty}{y^2})$$

$$\boxed{c_1 = p_1}$$

Then; $\boxed{p = -2\rho U^2 f^2(y) + p_1}$

e) x-momentum → Diff. eq for f.

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}}\right) = \cancel{-\frac{\partial p}{\partial x}} + \cancel{\mu\nabla^2 u} + \cancel{\rho b_x}$$

$$\rho\left(2Ux f'(y)\cdot 2Uf'(y) + (-2Uf(y)\cdot 2Ux f''(y)\right) = 0$$

Eliminating terms:

$$\rho\left(4U^2 f'(y)^2 \cancel{x} - 4U^2 f(y)\cdot f''(y)\cancel{x} = 0\right.$$

$$\boxed{f'(y)^2 - f(y)\cdot f''(y) = 0}$$

→ Regarding part c):

$$f(0) = f'(0) = 0$$

$$\downarrow$$

$$\underbrace{f'(0)^2}_{0} - \underbrace{f(0)}_{0}\cdot f''(y) = 0$$

✓

Fullfills !

(2.) Kármán - Pohlhausen approx.

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2 \quad \& \quad \underline{BC} \quad u = 0 \text{ at } y = 0$$

$$u = U, \ \frac{\partial u}{\partial y} = 0 \text{ at } y = \delta$$

1

$\boxed{0 = a}$

2 $u = U \to y = \delta$

$\frac{U}{U} = \boxed{b + c = 1}$

3/ $\frac{\partial u}{dy} = 0$ at $y = \delta$

$$\frac{du}{dy} = U\left(\frac{b}{\delta} + c\,\frac{2y}{\delta^2}\right)\Bigg|_{y=\delta} = 0 \to \frac{b}{\delta} + \frac{2c}{\delta} = 0$$

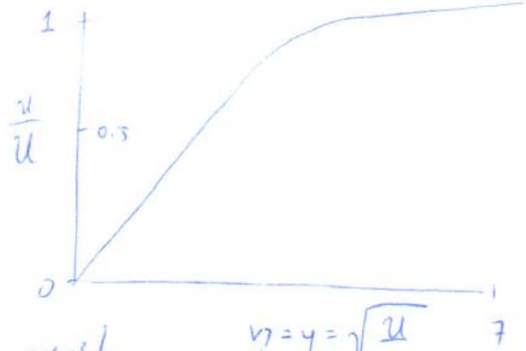
$\boxed{b + 2c = 0}$

Then: $\boxed{\begin{matrix} a = 0 \\ b = 2 \\ c = -1 \end{matrix}}$ $\boxed{\frac{u}{U} = \frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2}$

Blasius comparison:

$$f(\eta) = \frac{\psi}{\sqrt{v \cdot x U}} \quad \frac{u}{U} = f'(\eta) \quad \eta = \frac{y}{\sqrt{\frac{vx}{U}}}$$

$$u = Uf' \quad v = \frac{1}{2}\sqrt{\frac{vU}{x}}\,\eta f' - \frac{1}{2}\sqrt{\frac{vU}{x}}\,f$$



We apply Karman when exact solutions do not exist. (Blasius)

$\eta = y = \sqrt{\frac{U}{vx}}$

Cubic:

$$u(\eta) = U_\infty\left(A_0 + A_1\eta + A_2\eta^2 + A_3\eta_3\right)$$

What we do with Karman Polhausen (that can have more profiles than quadratic, cubic, ... is aproximating the blasius exact one to make things easier or when an exact solution can not be found.

So blasius solution is the better one between the three, followed by cubic and the quadratic karman one.