UNIVERSITAT POLITÈCNICA DE CATALUNYA

HomeWork 3

Fluid Mechanics

Group Members:

Waleed Ahmad Mirza

Xavi Corbella Coll

1)

a)

Solution:

<u>Dimensionless Analysis:</u>

Qualitative physics of the problem can be explained with the following equation.

$$\frac{\Delta P}{L} = f(\rho, v_o, R, R_1, \mu_1, \mu_2, \sigma)$$

Here the total no. of parameters are 8. In order to reduce the no. of parameters, they will be grouped in the form of non-dimensional constants.

The number of primary variables is r =3. In this analysis, the primary variables are ρ, v_o and R.

Total no. of constants to be found is n – r = 5

Now each parameter will be non-dimensionalized step by step.

For $\frac{\Delta P}{L}$:

$$M^o L^o T^o = M^{-1} L^{-2} T^{-2}. M^a L^{-3a}. L^b T^{-b}. L^c$$

Equation exponents of the primary dimensions

$$0 = 1 + a \qquad \Rightarrow a = -1$$

$$0 = -2 - 3a + b + c \qquad \Rightarrow c = +1$$

$$0 = -2 - b \qquad \Rightarrow b = -2$$

Hence $\pi_1 = \frac{\Delta P}{L} \frac{R_1}{\rho \overline{v}_o^2}$

By following a similar procedure for the rest of parameters i.e. $\mu_1,\ \mu_2, R$ and σ the following constants are obtained:

$$\Pi_2 = \frac{\mu_1}{\rho \overline{v}_o R_1}; \Pi_3 = \frac{\mu_2}{\rho \overline{v}_o R_1}; \Pi_4 = \frac{R}{R_1}; \Pi_5 = \frac{\sigma}{\rho \overline{v}_o{}^2 R_1}$$

Now the given problem can be analyzed by the following equation, which contains less parameters than the original equation:

$$\frac{\Delta P}{L} \frac{R_1}{\rho \overline{v}_o^2} = F\left(\frac{\mu_1}{\rho \overline{v}_o R_1}, \frac{\mu_2}{\rho \overline{v}_o R_1}, \frac{R}{R_1}, \frac{\sigma}{\rho \overline{v}_o{}^2 R_1}\right) \qquad (1)$$

b)

Solution:

In equation (1), there are several dimensionless numbers which are known:

- $\frac{\mu_1}{\rho \overline{v_o} R_1} = \frac{1}{Re}$ and $\frac{\mu_2}{\rho \overline{v_o} R_1} = \frac{1}{Re}$, where Re is the Reynolds number.
- $\frac{\sigma}{\rho \overline{v_o}^2 R_1} = \frac{1}{We}$ where We is the Webber number.
- $\frac{\Delta P}{L} \frac{R_1}{\rho \overline{\overline{v}_o^2}}$ can be seen as a friction factor

In this problem, waves could be formed at the interface depending on the effects of gravity, surface tension and fluid inertia.Since gravity effects have been neglected, the presence of waves will be related to surface tension and fluid inertia, which can be related by Weber number $We = \frac{\rho \overline{v_o}^2 R_1}{\sigma}$.

Under flowing conditions, surface tension is the force that opposes the increase in surface area caused by deformation. In order to minimize having waves and other non-desired effects, the surface tension must be greater than fluids inertia:

$$\frac{\rho \overline{v_o}^2 R_1}{\sigma} < 1$$

To completly avoid having waves, surface tension must be much greater than inertia:

$$\frac{\rho \overline{v_o}^2 R_1}{\sigma} \ll 1$$

c)

Solution:

The formation of waves due to gravity depends on the so-called "reduced gravity":

$$g' = g \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$$

In this case, $\rho_2 = \rho_1 \rightarrow g' = 0$. So, gravity effects can be neglected.

Another way to see that is to calculate the forces due to gravity. The force acting on a fluid which is inside another fluid will be:

$$F = B - W = Vg(\rho_2 - \rho_1)$$

So, the net force is 0 and gravity effects can be neglected.

d)

Solution:

The Navier stokes equation for incompressible, isothermal Newtonion flow with flow velocity $\vec{v} = (0,0,v_z(r))$ are:

r-component

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \rho g_r + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial v^2}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

θ-component

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z-component

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

The r and θ components of momentum equation prove themselves, as well as the continuity equation.

For z component, the Navier Stokes Equation is as follows:

$$\frac{-\partial P}{\partial z} + \mu_1\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial v_z}{\partial r}\right)\right] = 0 \qquad 0 \le r \le R_1$$

$$-\frac{\partial P}{\partial z} + \mu_2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\partial v_z}{\partial r}\right)\right] = 0 \qquad R_1 \le r \le R$$

Appropriate boundary condition for the problems are:

- For r=R:

$$v_z(R) = 0$$

- For r=R_1

$$v(R_1^-) = v(R_1^+) \coloneqq v_0$$

$$\mu_1 \left.\frac{\partial v_z}{\partial r}\right|_{R_1^-} = \mu_2 \left.\frac{\partial v_z}{\partial r}\right|_{R_1^+}$$

e)

Solution:

Integrating:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = -\frac{1}{\mu}\frac{\Delta P}{L}$$

$$r\frac{\partial v_z}{\partial r} = -\frac{1}{\mu}\frac{\Delta P}{L}\frac{r^2}{2} + A_1$$

$$v_z(r) = -\frac{1}{\mu}\frac{\Delta P}{L}\frac{r^2}{4} + Aln(r) + B$$

Thus:

$$v_z^{(1)}(r) = -\frac{1}{\mu_1}\frac{\Delta P}{L}\frac{r^2}{4} + A_1 ln(r) + B_1 \qquad 0 \le r \le R_1$$

$$v_z^{(2)}(r) = -\frac{1}{\mu_2}\frac{\Delta P}{L}\frac{r^2}{4} + A_2 ln(r) + B_2 \qquad R_1 \le r \le R$$

The constant A_1 must vanish to avoid computing ln(0). After applying the boundary conditions stated before the following equations are obtained:

$$v_z^{(1)}(r) = \frac{1}{4\mu_1}\frac{\Delta P}{L}(R_1^2 - r^2) + \frac{1}{4\mu_2}\frac{\Delta P}{L}(R^2 - R_1^2) \qquad 0 \le r \le R_1$$

$$v_z^{(2)} = \frac{1}{4\mu_2}\frac{\Delta P}{L}(R^2 - r^2) \qquad R_1 \le r \le R$$

At the interface:

$$v_o = v_z^{(1)}(R_1) = v_z^{(2)}(R_1) = \frac{1}{4\mu_2}\frac{\Delta P}{L}(R^2 - R_1^2)$$

Shear stress is estimated as follow:

$$\tau_{rz}^{(1)} = \mu_2\frac{\partial v_z^{(1)}}{\partial r} = -\frac{r}{2}\frac{\Delta P}{L} \qquad 0 \le r \le R_1$$

$$\tau_{rz}^{(2)} = \mu_2\frac{\partial v_z^{(2)}}{\partial r} = -\frac{r}{2}\frac{\Delta P}{L} \qquad R_1 \le r \le R$$

The radial distribution of velocity and shear-stress is plotted in Fig. 1

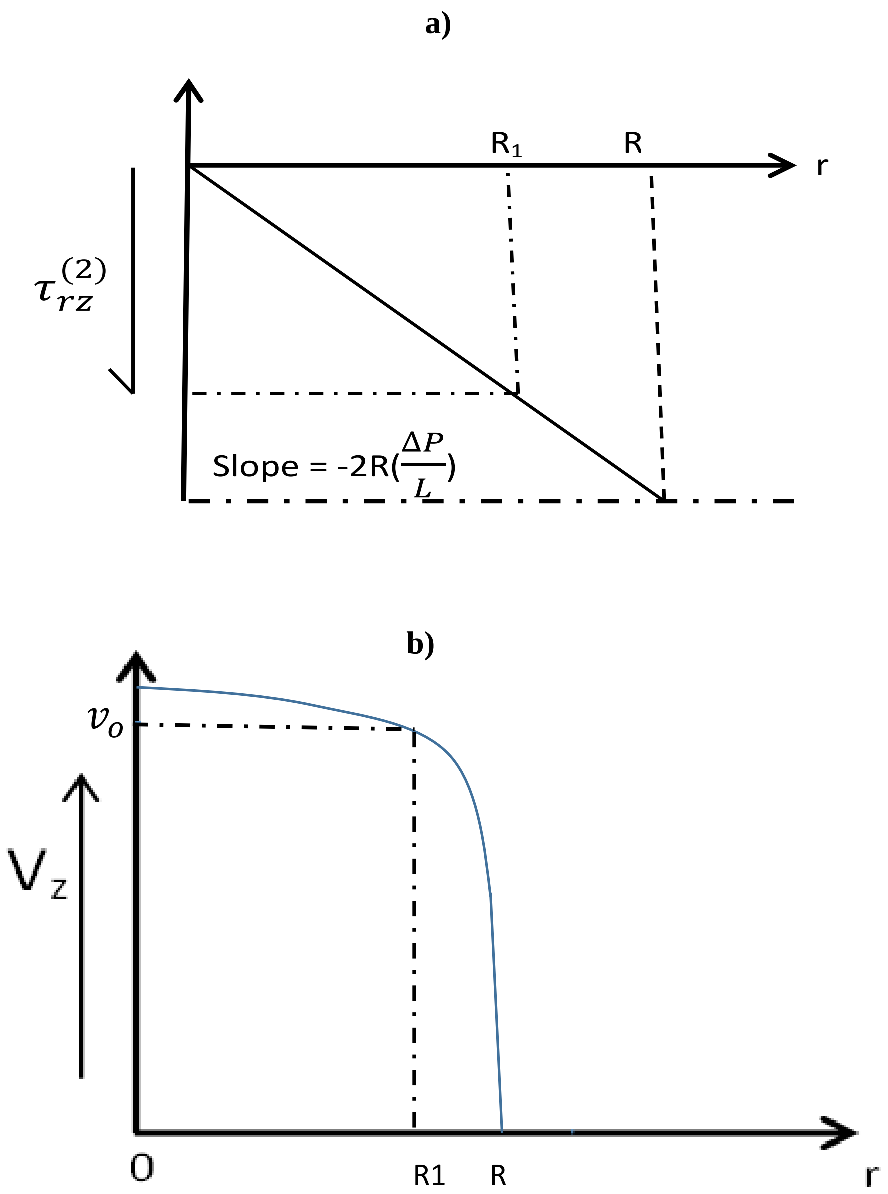

Fig. 1- a) Velocity and b) shear stress distributions.

f)

Solution:

Volume flow rate of oil is as follow:

$$Q_o = \int_o^{R_1} v_z^{(1)} 2\pi r dr$$

$$Q_o = \pi \frac{\Delta P}{L} \int_o^{R_1} \left(\frac{r}{\mu_1} [R_1^2 - r^2] + \frac{r}{\mu_2} [R^2 - R_1^2] \right) dr$$

$$= \frac{\pi}{2} \frac{\Delta P}{L} \left(\int_o^{R_1} \frac{r}{\mu_1} [R_1^2 - r^2] dr + \int_o^{R_1} \frac{r}{\mu_2} [R^2 - R_1^2] dr \right)$$

$$= \frac{\pi}{2} \frac{\Delta P}{L} \left(\frac{R_1^4}{2\mu_1} - \frac{R_1^4}{4\mu_1} + \frac{R_1^2}{2\mu_2} [R^2 - R_1^2] \right)$$

$$= \frac{\pi}{2} \frac{\Delta P}{L} \left(\frac{R_1^4}{4\mu_1} + \frac{R_1^2 R^2}{2\mu_2} - \frac{R_1^4}{2\mu_2} \right)$$

$$Q_o = \frac{\pi}{4\mu_2} \left(\frac{\Delta P}{L} \right) R_1^2 [R^2 - R_1^2] + \frac{\pi}{8\mu_1} \left(\frac{\Delta P}{L} \right) R_1^4$$

Similarly for the water,

$$Q_w = \int_{R_1}^{R} v_z^{(2)} 2\pi r dr = \frac{\pi}{8\mu_2} \left(\frac{\Delta P}{L} \right) [R_1^2 - R^2]$$

2)

The shock tube is depicted in **Fig. 2**, and the initial values of pressure are depicted in **Fig. 3**.

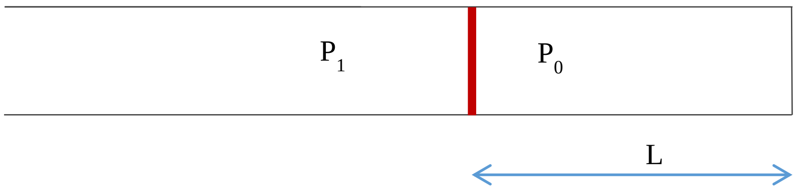

Fig. 2-Shock tube before breaking the diaphragm.

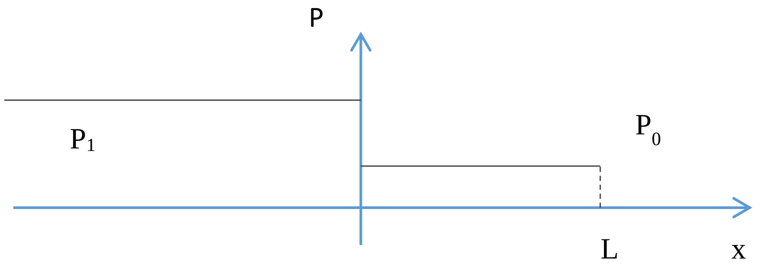

Fig. 3-Initial conditions.

The diaphragm breaks at t= 0 producing a discontinuity at x=0. Since $\frac{P_1 - P_0}{P_0} \ll 1$, the disturbance is small and the problem can be solved assuming acoustic waves.

When the diaphragm burst at t=0, pressure waves are released to equalize the pressure. This results in an expansion wave moving from x=0 to the left, and a compression wave moving to the right. The space outside the wave influenced regions ($x < -ct$ and $x > ct$) will retain thier initial state. However both pressure and velocity will change , in the space influenced by the waves.

The new values of pressure and velocity can be calculated using the characteristic lines:

$$\begin{cases} \frac{u}{c} + \frac{1}{\gamma}\frac{p}{p_0} = constant & \text{along} \quad x - ct = constant \\ \frac{u}{c} - \frac{1}{\gamma}\frac{p}{p_0} = constant & \text{along} \quad x + ct = constant \end{cases}$$

Using the characteristic lines for a point affected by the waves (see Fig. 4):

$$\frac{u_2}{c} + \frac{1}{\gamma}\frac{p_2}{p_0} = \frac{1}{\gamma}\frac{p_0}{p_0}$$

$$\frac{u_2}{c} - \frac{1}{\gamma}\frac{p_2}{p_0} = -\frac{1}{\gamma}\frac{p_1}{p_0}$$

Thus:

$$\begin{cases} p_2 = \frac{1}{2}(p_1 + p_0) \\ u_2 = \frac{c}{2\gamma}\left(\frac{p_1}{p_0} - 1\right) \end{cases}$$

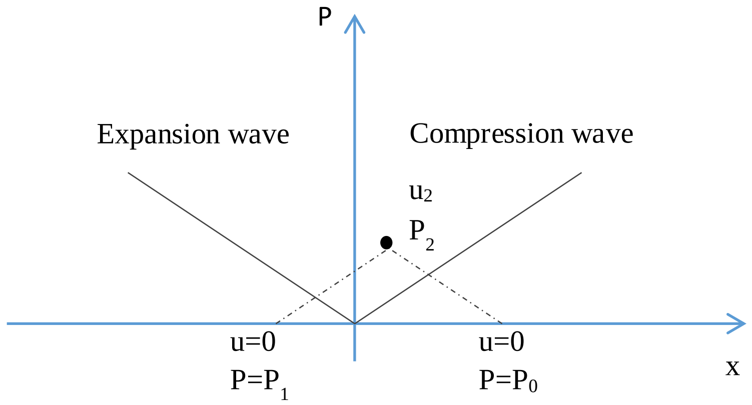

Fig. 4

At $t = \frac{L}{c}$ the compression wave travelling to the right collide the wall. In order to avoid having a non-zero velocity normal to the wall, a reflected wave travels to the left counteracting the velocity due to the compression wave (Fig. 5).

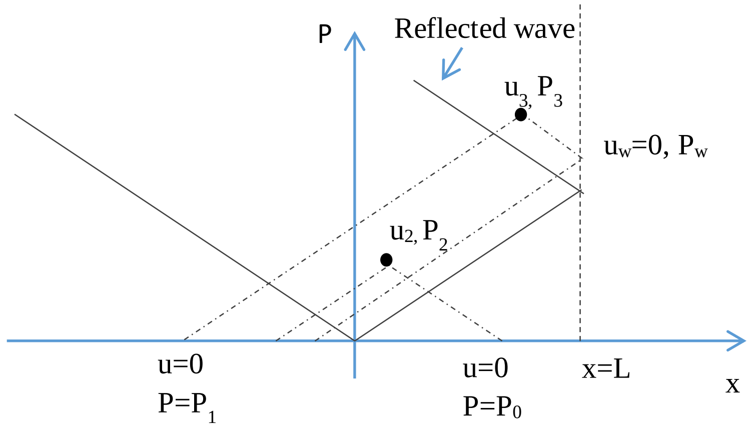

Fig. 5

The pressure at the wall after the reflected wave can be found analyzing the Riemann invariants and taking into account that $u_w = 0$:

$$\frac{u_w}{c} + \frac{1}{\gamma}\frac{p_w}{p_0} = \frac{1}{\gamma}\frac{p_1}{p_0} \rightarrow p_w = p_1$$

P_3 and u_3 can be obtained using the characteristic lines:

$$\frac{u_3}{c} + \frac{1}{\gamma}\frac{p_3}{p_0} = \frac{1}{\gamma}\frac{p_1}{p_0}$$

$$\frac{u_3}{c} - \frac{1}{\gamma}\frac{p_3}{p_0} = -\frac{1}{\gamma}\frac{p_w}{p_0} = -\frac{1}{\gamma}\frac{p_1}{p_0}$$

Thus:

$$\begin{cases} p_3 = p_1 \\ u_3 = 0 \end{cases}$$

A summary of the distributions of pressure and velocity in the shock tube for different values of time is depicted in Fig. 6 and Fig. 7.

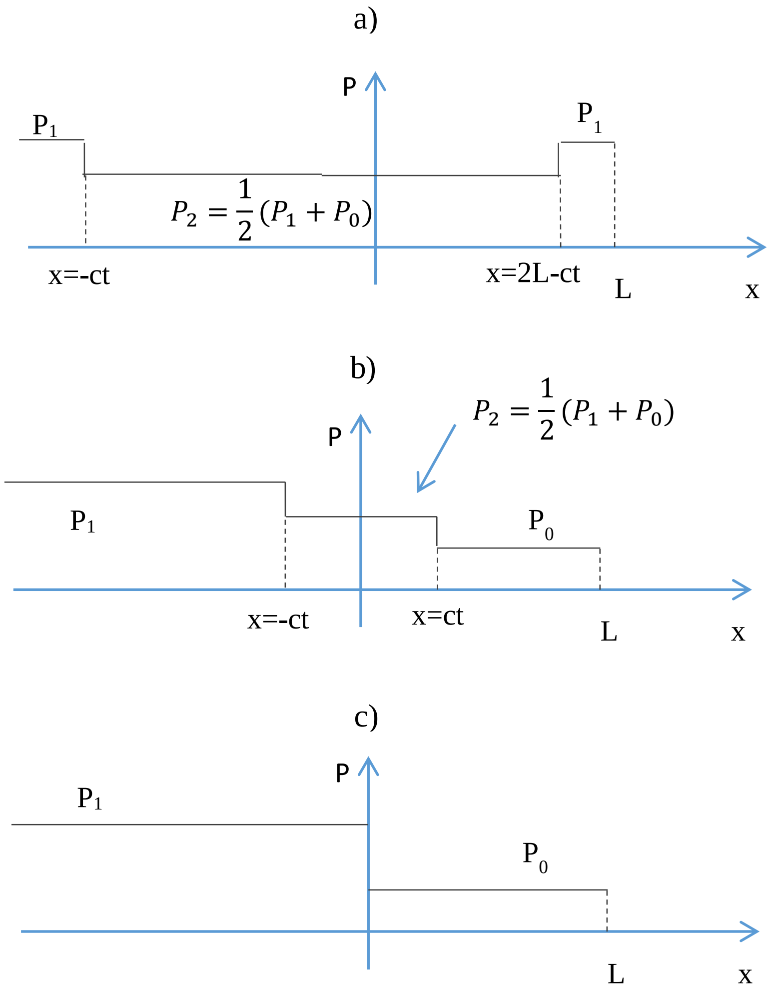

Fig. 6-Distribution of pressure at different times: a) $t > \frac{L}{c}$ b) $0 < t < \frac{L}{c}$ c) $t < 0$

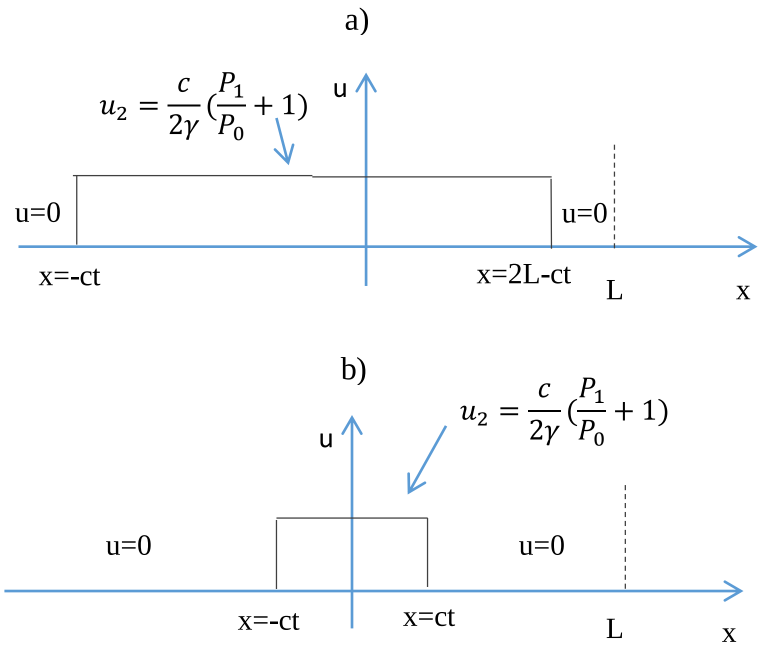

Fig. 7- Distribution of velocity at different times: a)$\boldsymbol{t > \frac{L}{c}}$ b) $\boldsymbol{0 < t < \frac{L}{c}}$