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MSc COMPUTATIONAL MECHANICS

A.F.M. :- Homework 3 (1)

Q1 →

(a) $$\frac{\Delta P}{L} = f(\rho, \bar{v}_0, R, R_1, \mu_1, \mu_2, \sigma)$$

	$\Delta P/L$	ρ	$\bar{v}_0$	R	R_1	μ_1	μ_2	σ
M	1	1	0	0	0	1	1	1
L	-2	-3	1	1	1	-1	-1	0
T	-2	0	-1	0	0	-1	-1	-2

rank of matrix = 3
no. of π (dimension-less quantities) = 8 - 3 = 5

three related variables → $\rho, \bar{v}_0, R$

$$\pi_1 = \frac{\Delta P}{L} \rho^a v_0^b R^c$$

$$M^0L^0T^0 = (M^1L^{-2}T^{-2})(M^1L^{-3})^a(L^1T^{-1})^b(L^1)^c$$

$$M^0L^0T^0 = M^{(1+a)} L^{(-2-3a+b+c)} T^{(-2-b)}$$

$1 + a = 0$; $-2 - 3a + b + c = 0$; $-2 - b = 0$

$\boxed{a = -1}$; $-2 - 3(-1) + (-2) + c = 0$; $\boxed{b = -2}$

$-2 + 3 - 2 + c = 0$

$\boxed{c = 1}$

$$\pi_1 = \frac{\Delta P}{L}\left(\frac{1}{\rho}\right)\left(\frac{1}{\bar{v}_0^2}\right)(R) \quad ; \quad \boxed{\pi_1 = \frac{\Delta P\, R}{L \rho \bar{v}_0^2}}$$

$$\pi_2 = R_1\, \rho^a\, \bar{V}_0^b\, R^c$$

$$M^0 L^0 T^0 = (L^1)(M^1L^{-3})^a (L^1T^{-1})^b (L^1)^c$$

$$M^0 L^0 T^0 = L^{(1-3a+b+c)}\ M^{(a)}\ T^{(-b)}$$

$\boxed{a=0}$; $-b=0$; $1-3a+b+c=0$

$\boxed{b=0}$; $1+c=0$

$\boxed{c=-1}$

$$\pi_2 = R_1\, \rho^0\, \bar{V}_0^0\, R^{-1} \quad ; \quad \boxed{\pi_2 = \frac{R_1}{R}}$$

$$\pi_3 = \mu_1\, \rho^a\, \bar{V}_0^b\, R^c$$

$$M^0 L^0 T^0 = (M^1L^{-1}T^{-1})(M^1L^{-3})^a (L^1T^{-1})^b (L^1)^c$$

$$M^0 L^0 T^0 = M^{(1+a)}\ L^{(-1-3a+b+c)}\ T^{(-1-b)}$$

$1+a=0$; $-1-3a+b+c=0$; $-1-b=0$

$\boxed{a=-1}$; $-1+3-1+c=0$; $\boxed{b=-1}$

$\boxed{c=-1}$

$$\pi_3 = \mu_1\, \rho^{-1}\, \bar{V}_0^{-1}\, R^{-1} \Rightarrow \boxed{\frac{\mu_1}{\rho\, \bar{V}_0\, R} = \pi_3}$$

$$\pi_4 = \mu_2\, \rho^a\, \bar{V}_0^b\, R^c$$

$$M^0 L^0 T^0 = (M^1L^{-1}T^{-1})(M^1L^{-3})^a (L^1T^{-1})^b (L^1)^c$$

$\Rightarrow \boxed{a=-1}$; $\boxed{b=-1}$; $\boxed{c=-1}$

$$\therefore \boxed{\pi_4 = \frac{\mu_2}{\rho\, \bar{V}_0\, R}}$$

$$\pi_5 = \sigma \, \rho^a \bar{V}_0^b R^c$$

$$M^0 L^0 T^0 = M^1 T^{-2} (M^1 L^{-3})^a (L^1 T^{-1})^b (L^1)^c$$

$$M^0 L^0 T^0 = M^{(1+a)} L^{(-3a+b+c)} T^{(-2-b)}$$

$1+a=0$; $-3a+b+c=0$; $-2-b=0$

$\boxed{a=-1}$; $-3(-1)+(-2)+c=0$; $\boxed{b=-2}$

$3-2+c=0$

$\boxed{c=-1}$

$$\pi_5 = \sigma\left(\frac{1}{\rho}\right)\left(\frac{1}{\bar{V}_0^2}\right)\left(\frac{1}{R}\right) ; \boxed{\pi_5 = \frac{\sigma}{\rho \bar{V}_0^2 R}}$$

Following are the five dimensionless quantities obtained

$$\pi_1 = \frac{\Delta P R}{L \rho \bar{V}_0^2}$$

$$\pi_2 = \frac{R_1}{R}$$

$$\pi_3 = \frac{\mu_1}{\rho \bar{V}_0 R}$$

$$\pi_4 = \frac{\mu_2}{\rho \bar{V}_0 R}$$

$$\pi_5 = \frac{\sigma}{\rho \bar{V}_0^2 R}$$

(b) π_3 and π_4 correponds to Reynold's Number π_5 correponds to Weber number. All of these Weber number is important as it tells weather waves will develop.

Interfacial tension is the work force which must be expanded to increase size of the interface between 2 adjacent phases which donot mix completely with one another. That means if no external forces apply, the liquid phases minimize the size of their interface. Therefore waves will not form unless inertia forces are very small compared to surface tension which means if the Weber no. is very small.

$$\text{Weber no.} << 1$$

$$\Rightarrow \pi_5 >> 1$$

$$\Rightarrow \frac{\sigma}{R} >> \rho \bar{v}_0^2$$

(c) Since the fluids have same density $\rho_1 = \rho_2 = \rho$; gravity effects can be neglected and it is resonable to assume that gravity will not contribute to the formation of waves at interface.

(d) $\bar{v} = (0, 0, v_z(r))$

Navier strokes eqn.:-

$$-\frac{\partial p}{\partial r} = 0$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = 0$$

$$-\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = 0$$

where $\mu = \begin{cases} \mu_1 & \text{if } 0 < r < R_1 \\ \mu_2 & \text{if } R_1 < r < R \end{cases}$

assuming the interface is cylindrical, the boundry condition at the interface

$$v_z\Big|_{r=R} = 0$$

for no-slip : $\tau_{rz}\Big|_{r=R_1^{(-)}} = \tau_{rz}\Big|_{r=R_1^{(+)}}$

$$\Rightarrow \mu_1 \frac{\partial v_z}{\partial r}\Big|_{r=R_1^{(-)}} = \mu_2 \frac{\partial v_z}{\partial r}\Big|_{r=R_1^{(+)}}$$

also $\frac{\partial v_z}{\partial r}\Big|_{r=0} = 0$ (due to geometry)

(c) considering the z component of Navier-Strokes eq^n, we know the solutions are of form $\frac{\partial P}{\partial z} = \frac{\mu}{r}\left(r \frac{\partial v_z}{\partial r}\right)$

$$\Rightarrow \frac{\Delta P}{\mu L} \cdot r = \frac{\partial}{\partial r}\left(r \frac{\partial z}{\partial r}\right)$$

$$v_z = \frac{1}{4\mu_1} \frac{\Delta P r^2}{L} + a_1 \ln(r) + b_1 \quad (0 < r < R_1)$$

&

$$v_z = \frac{1}{4\mu_2} \frac{\Delta P r^2}{L} + a_2 \ln(r) + b_2 \quad (R < r < R)$$

Using B.C.

$$\frac{\partial v_z}{\partial r}\Big|_{r=0} = 0 \quad \Rightarrow a_1 = 0$$

$$\mu \frac{\partial v_z}{\partial r}\Big|_{r=R_1^{(-)}} = \mu_2 \frac{\partial v_z}{\partial r}\Big|_{r=R^{(+)}} \Rightarrow a_2 = 0$$

$v_z\Big|_{r=R} = 0$ gives $b_2 = -\frac{1}{4\mu_2} \frac{\Delta P R^2}{L}$

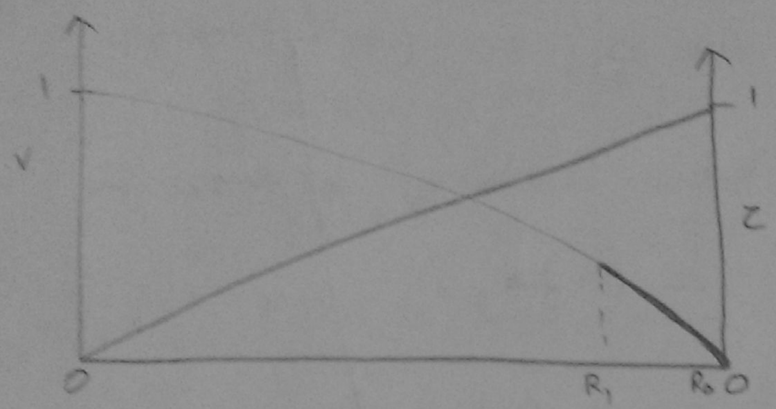
& $v_z\Big|_{r=R_1^{(-)}} = v_z\Big|_{r=R_1^{(+)}} \Rightarrow b_1 = \frac{-1}{4\mu_1} \frac{\Delta P(R_1^2 - r^2)}{L} - \frac{1}{4\mu_2} \frac{\Delta P(R^2 - R_1^2)}{L}$

$$v_z = \frac{1}{4\mu_2} \frac{\Delta P}{L}(R^2 - r^2) \quad \text{if } (R_1 < r < R)$$

& $$v_z = \frac{1}{4\mu_1} \frac{\Delta P}{L}(R_1^2 - r^2) + \frac{1}{4\mu_2} \frac{\Delta P(R^2 - R_1^2)}{L} \quad \text{if } 0 < r < R_1$$

$$v_z\Big|_{\text{interface}} = \frac{1}{4\mu_2}\frac{\Delta P}{L}\left(R - R_1^2\right)$$

$$\tau_{rz} = \mu\frac{\partial v_z}{\partial r} = -\frac{1}{2}\frac{\Delta P}{L}r$$



(f)

$$Q_o = \int_0^{R_1} 2\pi r v_z\,dr = \frac{2\pi}{4}\left(\frac{\Delta P}{L}\right)\int_0^{R_1}\frac{r}{\mu_1}\left(R_1^2 - r^2\right) + \frac{r}{\mu_2}\left(R^2 - R_1^2\right)dr$$

$$= \frac{\pi}{8\mu_1}\frac{\Delta P}{L}\cdot R_1^4 + \frac{\pi}{4\mu_2}\frac{\Delta P}{L}\left(R^2 - R_1^2\right)R_1^2$$

$$Q_w = \int_{R_1}^{R} 2\pi r v_z\,dr = \int_{R_1}^{R}\frac{2\pi\,\Delta P}{4\mu_2 L}\left(R^2 - r^2\right)r\,dr$$

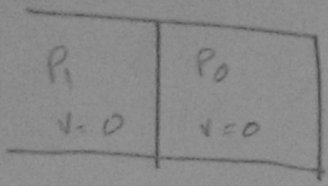
$$= \frac{\pi}{2\mu_2}\frac{\Delta P}{L}\int_{R_1}^{R}\left(R^2 r - r^3\right)dr = \frac{\pi}{2\mu_2}\frac{\Delta P}{L}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_{R_1}^{R}$$

$$= \frac{\pi}{2\mu_2}\frac{\Delta P}{L}\left[\left(\frac{R^4}{4}\right) - \left(\frac{R^2 R_1^2}{2} - \frac{R_1^4}{4}\right)\right]$$

$$= \frac{\pi}{8\mu_2}\frac{\Delta P}{L}\left[R^4 - 2R_1^2 R^2 + R_1^4\right]$$

$$Q_w = \frac{\pi}{8\mu_2}\frac{\Delta P}{L}\left(R^2 - R_1^2\right)^2$$

Q2.



$P_1 = P_0 + p'$

$P_0 >> p'$

$$\frac{(P_1 - P_0)}{P_0} << 1$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = 0$$

$$\frac{\partial}{\partial t}(\rho \bar{v}) + \nabla \cdot (\rho \bar{v} \otimes \bar{v} + P I) = \rho \bar{b}$$

$$\frac{\partial}{\partial t}(\rho E) + \nabla \cdot ((\rho E + P)\bar{v}) = \bar{v} \cdot \rho \bar{b}$$

$$\rho = \rho_0 + \rho'$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho_0 u) + \cancel{\frac{\partial}{\partial x}(\rho' u)}^{\,0}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0 \qquad — \text{(A)}$$

$$P = P_0 + p'$$

(neglecting $v \otimes v$ and body forces)

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} = 0 \qquad — \text{(B)}$$

let $\rho_n = \dfrac{\rho - \rho_0}{\rho_0} \Rightarrow \rho = \rho_0(1 + \rho_n)$

$$\frac{\partial \rho}{\partial t} = \rho_0 \frac{\partial \rho_n}{\partial t}$$

$$\text{eq}^n \text{ (A)} \longrightarrow \frac{\partial \rho_n}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad — \text{(C)}$$

for ideal gas $\dfrac{P}{\rho^n} = \dfrac{P_0}{\rho_0^{\,n}}$ where $n = \dfrac{C_P}{C_v}$

$$\Rightarrow \frac{P}{(1+\rho_n)^n \rho_0^{\,n}} = \frac{P_0}{\rho_0^{\,n}} \Rightarrow P = P_0 (1 + \rho_n)^n$$

$$P \simeq P_0 (1 + n \rho_n)$$

$$\Rightarrow \frac{\partial P}{\partial x} = n P_0 \frac{\partial \rho_n}{\partial x}$$

eqⁿ (B) $\longrightarrow \rho_0 \frac{\partial u}{\partial t} + \gamma p_0 \frac{\partial \rho_n}{\partial x} = 0$

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{\gamma p_0}{\rho_0} \frac{\partial \rho_n}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} + c^2 \frac{\partial \rho_n}{\partial x} = 0 \quad \text{where } c^2 = \frac{\gamma p_0}{\rho_0}$$

$\therefore$ we have to solve following system of eqⁿ

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad — \quad (D)$$

$$\frac{\partial u}{\partial t} + c^2 \frac{\partial \rho}{\partial x} = 0 \quad — \quad (E)$$ (also replacing ρ_n by ρ)

eqⁿ (D) + (E) / c

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} + \frac{1}{c}\frac{\partial u}{\partial t} + c\frac{\partial \rho}{\partial x} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{u}{c} + \rho\right) + c\frac{\partial}{\partial x}\left(\frac{u}{c} + \rho\right) = 0$$

let $z = \frac{u}{c} + \rho$

$$\therefore \frac{\partial z}{\partial t} + c\frac{\partial z}{\partial x} = 0$$

eqⁿ (D) − (E)/c gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} - c\frac{\partial \rho}{\partial x} - \frac{\partial u}{c\,\partial t} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\rho - \frac{u}{c}\right) + c\frac{\partial}{\partial x}\left(\rho - \frac{u}{c}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{u}{c} - \rho\right) - c\frac{\partial}{\partial x}\left(\frac{u}{c} - \rho\right) = 0$$

let, $k = \frac{u}{c} - \rho$

$$\frac{\partial k}{\partial t} - c\frac{\partial k}{\partial x} = 0$$

the new system we have to solve is

$$\frac{\partial J}{\partial t} + c\frac{\partial J}{\partial x} = 0$$

$$\frac{\partial K}{\partial t} - c\frac{\partial K}{\partial x} = 0$$

using characteristic line method for eqn (A)

$$\frac{\partial J}{\partial t} + c\frac{\partial J}{\partial x} = 0$$

$$\frac{dx}{ds} = c \; ; \; \frac{dt}{ds} = 1$$

assume $t(0) = 0$

$$t = s$$

$$x = ct + b$$

$$\Rightarrow \frac{x}{c} - \frac{b}{c} = t$$

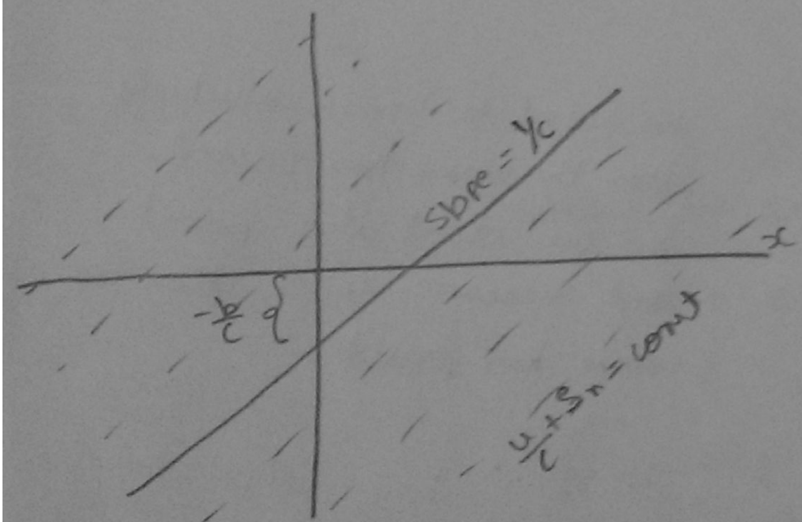
$$\text{Slope} = \frac{1}{c} \; (+ve)$$

$$\frac{dJ}{ds} = \frac{\partial J}{\partial x}\frac{dx}{ds} + \frac{\partial J}{\partial t}\cdot\frac{dt}{ds}$$

if $\frac{dx}{ds} = c$ & $\frac{dt}{ds} = 1$

we get

$$\frac{dJ}{ds} = \frac{\partial J}{\partial t} + c\frac{\partial J}{\partial x} = 0$$



now depending on value of b (initial condition), we will have family of lines shown by dotted lines.

using characteristic line method on eqn (B)

$$\frac{\partial K}{\partial t} - c\frac{\partial K}{\partial x} = 0$$

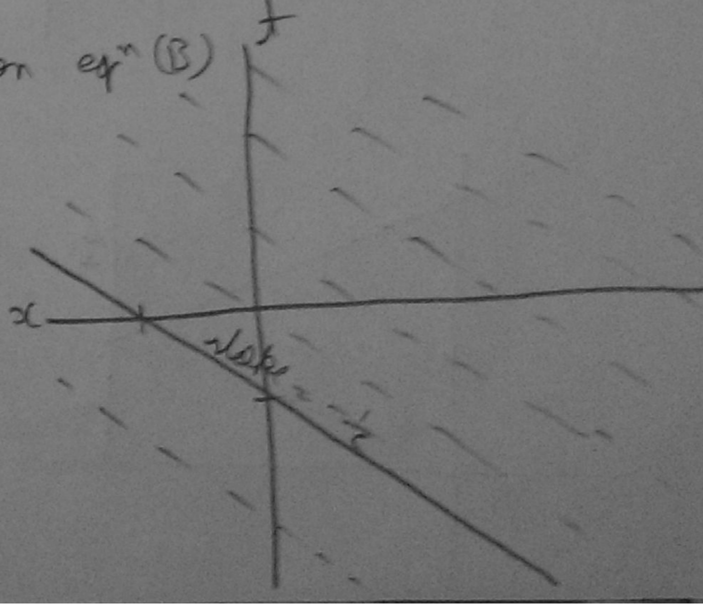
$$\frac{dx}{ds} = -c \; ; \; \frac{dt}{ds} = 1$$

$$t = s$$

(assume $t(0) = 0$)

$$x = -ct + d$$

$$-\frac{x}{c} - \frac{d}{c} = t$$



Slope = $-\frac{1}{c}$ (-ve)

we get multiple dotted lines based on initial condition

our problem is a combination of these two graphs

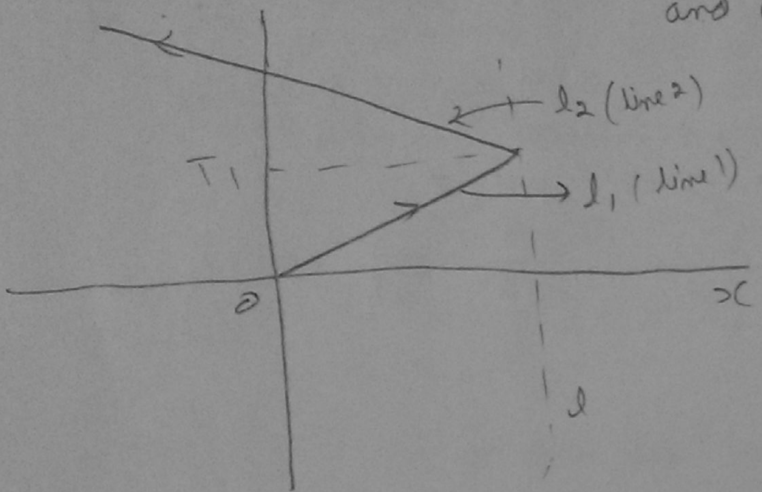
for time = 0 to T_1 forward wave

for time = T_1 to ∞ backward wave

also, at $t=0$; $x=0$ (initial condition)

at $t=T_1$; $x=x_1$ (wall)

and continuity and characteristic line.



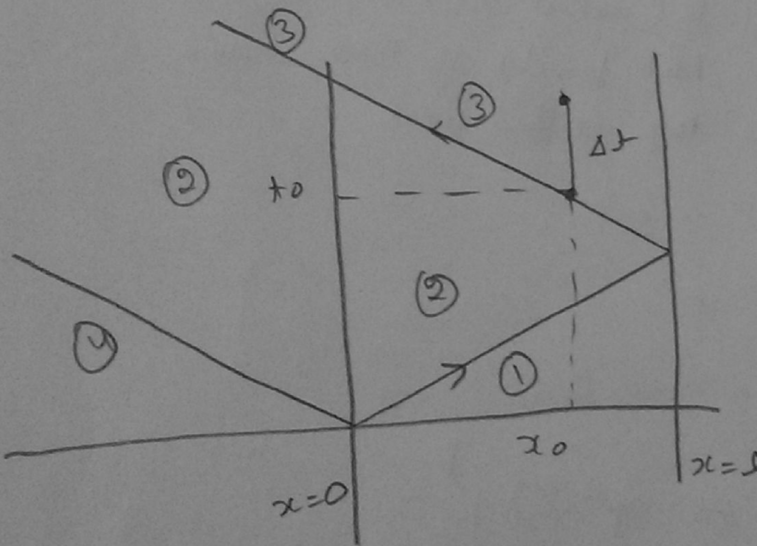
Line 1 eq^n :-

$$t = \frac{x}{c}$$

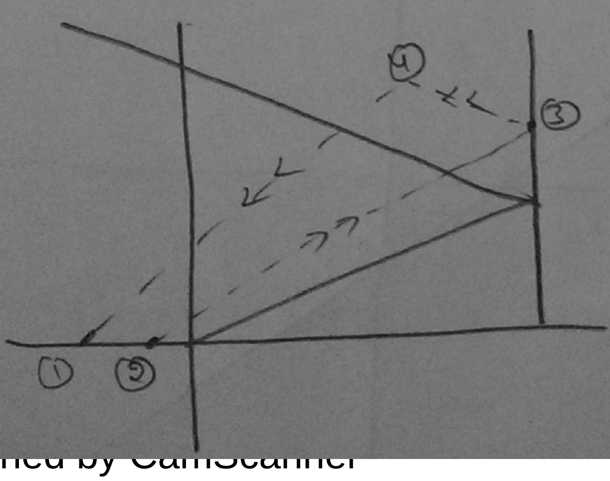
Line 2 eq^n :-

$$x = -c(t - T_1) + l$$

Lets draw the regions and interest



we want to know solution after the wave has passed. So we start at a point and measure u, ρ after the wave has passed.



1 - 4 $\frac{u}{c} + S_n$ = constant

3 - 4 $\frac{u}{c} - S_n$ = constant

2 - 3 $\frac{u}{c} + S_n$ = constant

at 1 $\rho = \rho_1$

$$\frac{u_1}{c} + \varsigma_1 = \frac{u_4}{c} + \varsigma_4$$

$$\boxed{\varsigma_1 = \frac{u_4}{c} + \varsigma_4} \quad \text{— (F)}$$

$$\frac{u_2}{c} + \varsigma_2 = \frac{u_3}{c} + \varsigma_3$$

$$\varsigma_2 = \frac{u_3}{c} + \varsigma_3 \qquad (u_2 = 0 \text{ at boundary})$$

$$\boxed{\varsigma_3 = \varsigma_2} = \varsigma_1$$

Between points 3 and 4

$$\frac{u_3}{c} - \varsigma_3 = \frac{u_4}{c} - \varsigma_4$$

$$\boxed{-\varsigma_1 = \frac{u_4}{c} - \varsigma_4} \quad \text{— (G)}$$

F and G are two equations with 2 unknowns.

$$\varsigma_1 = \frac{u_4}{c} + \varsigma_4$$

$$-\varsigma_1 = \frac{u_4}{c} - \varsigma_4$$

adding we get

$$0 = u_4 + 0$$

$$u_4 = 0$$

Subtracting

$$2\varsigma_1 = 2\varsigma_4$$

$$\varsigma_4 = \varsigma_1$$

$$\frac{P_4}{(\varsigma_4)^\gamma} = \frac{P_1}{(\varsigma_1)^\gamma}$$

$$P_4 = P_1$$

$\therefore$ just behind wave, $u = 0$ and $P = P_1$