Q1 → (c) $\nabla \cdot (F \times G) = G \cdot \nabla \times F - F \cdot \nabla \times G$

$$\nabla \cdot (F \times G) = \nabla \cdot (\varepsilon_{ijk} F_i G_j \hat{e}_k)$$

$$= \varepsilon_{ijk} \nabla \cdot (F_i G_j) \hat{e}_k$$

$$= \varepsilon_{ijk} (F_{i,k} G_j + F_i G_{j,k})$$

$$= \varepsilon_{ijk} F_{i,k} G_j + \varepsilon_{ijk} F_i G_{j,k}$$

$$= \varepsilon_{ijk} F_{i,k} G_j + \varepsilon_{ijk} F_i G_{j,k}$$

$$= \varepsilon_{jki} F_{i,k} G_j \delta_{jj} - \varepsilon_{ikj} G_{j,k} F_i \delta_{ii}$$

$$= G \cdot (\nabla \times F) - F \cdot \nabla \times G$$

(a) $\nabla \cdot (\nabla \times \vec{F}) = 0$

$$\nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot (\varepsilon_{ijk} F_{k,j} \hat{e}_i)$$

$$= \varepsilon_{ijk} F_{k,ji}$$

$$= \varepsilon_k \left(\frac{-\partial F_k}{\partial x_j \partial x_i} + \frac{\partial F_k}{\partial x_i \partial x_j} \right)$$

1 2 3
2 3 1

$$= \underline{\underline{0}}$$

(b) $\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \nabla^2 F$

$$\nabla \times (\nabla \times F) = \varepsilon_{ijk}\, \varepsilon_{klm}\, F_{m,lj}$$

$$= \varepsilon_{ijk}\, \varepsilon_{lmk}\, F_{m,lj}$$

$$= \left(\delta_{il}\,\delta_{jm} - \delta_{jl}\,\delta_{im}\right) F_{m,lj}$$

$$= \delta_{jm}\, F_{m,ij} - \delta_{im}\, F_{m,jj}$$

$$= F_{j,ij} - F_{i,jj} = \nabla(\nabla \cdot F) - \nabla^2 F$$

Solution 2 →

1^{st} law of thermodynamics

$$dQ = de + dW$$

$$dQ = de + PdV \quad - (1) \qquad (dw = Pdv)$$

2^{nd} law of thermodynamics

$$dQ = Tds \quad - (2) \qquad ds = \text{entropy}$$

from (1) & (2)

$$Tds = de + PdV$$

$$Tds = de + Pdv \quad (\text{for all infinitesimal changes})$$

$$Tds = de - \frac{P}{\rho^2}d\rho \qquad \left(\because v = \frac{1}{\rho},\ dv = \frac{-1}{\rho^2}.d\rho\right)$$

$$T\frac{Ds}{Dt} = \frac{De}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt}$$

$$\rho T\frac{Ds}{Dt} = \rho\frac{De}{Dt} - \frac{P}{\rho}\frac{D\rho}{Dt} \quad - (3)$$

$$\rho\frac{De}{Dt} = \sigma : \nabla v - \nabla . q$$

$$\rho\frac{De}{Dt} = -p\nabla.v - \nabla.q + \phi$$

where $\phi = \lambda(\nabla.v)^2 + 2\mu\nabla^2 v : \nabla v$

ϕ is always positive

form (3)

$$\rho T\frac{Ds}{Dt} = \left(-P\nabla.v - \nabla.q + \phi\right) - \frac{P}{\rho}\frac{D\rho}{Dt}$$

$$= \frac{-P}{\rho}\left(\frac{D\rho}{Dt} + \rho\nabla.v\right) - \nabla.q + \phi$$

→ 0 (continuity of mass eq^n)

$$\rho T\frac{Ds}{Dt} = -\nabla.q + \phi$$

$$\rho\frac{Ds}{Dt} = -\frac{\nabla.q}{T} + \frac{\phi}{T}$$

$$\nabla\cdot\left(\frac{q}{T}\right) = \frac{1}{T}\nabla\cdot q - \frac{q\cdot\nabla T}{T^2}$$

$$-\frac{\nabla\cdot q}{T} = -\frac{q\cdot\nabla T}{T^2} - \nabla\cdot\left(\frac{q}{T}\right)$$

$$\rho\frac{Ds}{Dt} = -\nabla\cdot\left(\frac{q}{T}\right) - \frac{q\,\nabla T}{T^2} + \frac{\phi}{T}$$

$$\rho\frac{Ds}{Dt} = -\nabla\cdot\left(\frac{q}{T}\right) + \underbrace{\frac{K\nabla T\cdot\nabla T}{T^2} + \frac{\phi}{T}}_{\Sigma} \qquad (q = -K\nabla T)$$

(where Σ is always positive)

$$\rho\frac{Ds}{Dt} = -\nabla\cdot\left(\frac{q}{T}\right) + \Sigma$$

$$\int_{V_t}\rho\frac{Ds}{Dt}\,dv = \int_{V_t}-\nabla\cdot\left(\frac{q}{T}\right)dV + \int_{V_t}\Sigma\,dv$$

using concept of reynolds lemma

$$\frac{D}{Dt}\int\rho s\,dV = \int_{V_t}-\nabla\cdot\left(\frac{q}{T}\right)dV + \int_{V_t}\Sigma\,dV$$

by Divergence theorem

$$\frac{D}{Dt}\int_{V_t}\rho s\,dV = -\int_{S_t}\frac{\bar{q}\cdot\bar{n}\,dS}{T} + \underbrace{\int_{V_t}\Sigma\,dV}$$

This is always positive

$$\boxed{\therefore\ \frac{D}{Dt}\int_{V_t}\rho s\,dV \geq -\int_{S_t}\frac{\bar{q}\cdot\bar{n}}{T}\,dS}$$

inequality always holds.