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HW3, Alba Navarro Casanova
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HW 3: Dimensional analysis, compressible flow and Navier-Stokes equations

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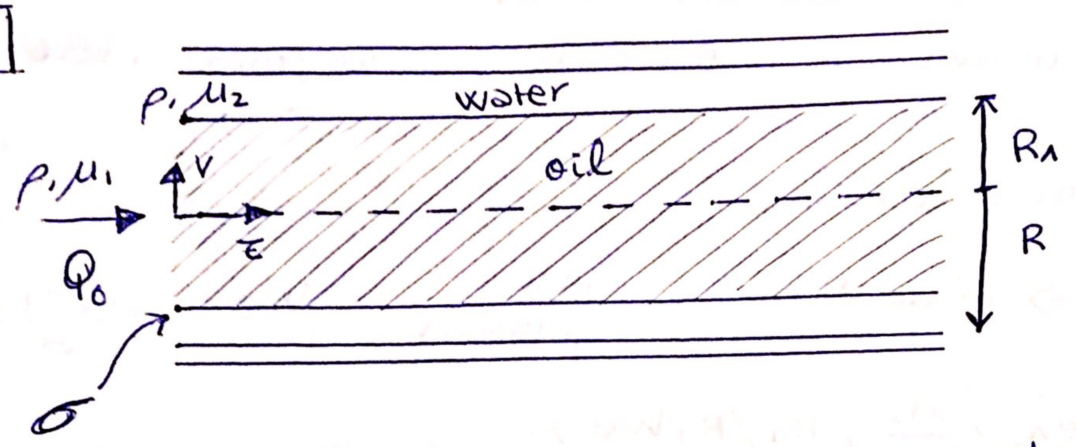


a) Use dimensional analysis to determine an appropiate dimensionless form for expressing the fully-developed pressure drop per unit length in the pipe $-\frac{\partial p}{\partial z} = \frac{\Delta P}{L}$ as a function of the relevant parameters in the problem:

$$\frac{\Delta P}{L} = f(\rho, \bar{V_0}, R, R_1, \mu_1, \mu_2, \sigma)$$

Using V_0, R_1 and ρ as our primary variables:

$$\bar{V_0} = \frac{Q_0}{\pi R_i^2}$$

$n = 8$
$r = 3$
$(n-r) = 5$ Pi groups

$\frac{\Delta P}{L}$ has units $\frac{P}{L} = \frac{F/A}{L} = \frac{N/m^2}{m} = \frac{\frac{kg\,m}{s^2 m^2}}{m} = \frac{kg}{m^2 s^2} =$

$= M^1 T^{-2} L^{-2}$ $\quad V = [L^1 T^{-1}]$ $\quad \rho = [M^1 L^{-3}]$

$R_1 = [L^1]$

$$M^1 T^{-2} L^{-2} = (L^1 T^{-1})^a (L^1)^b (M^1 L^{-3})^c$$

$c = 1$
$a = 2$
$b = -2-2+3 = -1$ $\Rightarrow \frac{V_0^2 \rho}{R_1}$

$$\frac{\Delta P/L}{(\rho \bar{V}_0^2/R_1)} = \phi\left(\frac{\mu_1}{\rho \bar{V}_0 R_1}, \frac{\mu_2}{\rho \bar{V}_0 R_1}, \frac{R}{R_1}, \frac{\sigma}{\rho \bar{V}_0^2 R_1}\right)$$

entonces los grupos adimensionales son por orden:

- un factor de fricción, nº de Reynolds para los fluidos interior y exterior
- $\frac{R}{R_1}$ proporción geométrica
- $\frac{\sigma}{\rho V_0^2 R_1}$ ⇒ nº de Weber $= \frac{\text{inercia}}{\text{capilaridad}}$ $\qquad We = \frac{\rho \bar{V}_0^2 R_1}{\sigma}$

$$f = \phi\left(Re_1, \frac{\mu_2}{\mu_1}, R_1/R, We\right)$$

b) Bajo condiciones de flujo, la tensión superficial es la que hace que no se formen superficies curvas. Entonces necesitamos que:

$$We = \rho \bar{V}_0^2 R_1/\sigma \ll 1$$

No es fácil de conseguir porque $\sigma \simeq 30 \times 10^{-3}\,N/m$ y $R_1 \simeq 1m$ necesitaremos $\bar{V}_0 \leq 5'5 mm/s$, aunque la pipa lubricada tolera olas interfaciales siempre y cuando no rompa

c) En el interfaz tenemos un esfuerzo cortante:

$$\tau_{rz}\big|_{r=R_1} = \tau_{rz}\big|_{r=R_1^*} \quad \text{o lo que es igual a}$$

$$\mu_1 \frac{\partial v_z}{\partial r}\bigg|_{r=R_1^-} = \mu_2 \frac{\partial v_z}{\partial r}\bigg|_{r=R_1^*}$$

La presión capilar aumenta através de la interfase pero es despreciable: si la presión capilar es pequeña comparada con el cambio hidroestático desde la parte superior a la inferior

$$p_1(z) = p_2(z) + \frac{\sigma}{R_1}$$

Tenemos:

$$\frac{\sigma}{R_1} \ll \rho g (2R_1) \qquad \frac{\sigma}{2\rho g R_1^2} \ll 1$$

entonces tenemos:

$$\frac{\partial p_2}{\partial z} = \frac{\partial p_1}{\partial z} = -\frac{\Delta P}{L} \Rightarrow \frac{\partial p}{\partial z} = -\frac{\Delta P}{L} \quad r \in [0, R]$$

d) Campo de velocidad (estado estacionario) $v = (0, 0, v_z(r))$.

y sustituyendo en Navier - Stokes eq:

$$0 = -\frac{\partial p}{\partial z} + \frac{\mu_1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \quad 0 \le r \le R_1$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\mu_2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \quad R_1 \le r \le R$$

Usando estas condi. de contorno junto con la condición de que hay una finita velocidad interfacial de la forma $v_i = v_z(r = R_1^-) = v_z(r = R_1^+)$ tenemos la siguiente expresión:

e)

$$v_z^{(1)} = \frac{1}{4\mu_1}\left(\frac{\Delta P}{L}\right)[R_1^2 - r^2] + \frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)[R^2 - R_1^2] =$$

$$= \frac{1}{4\mu_1}\left(\frac{\Delta P}{L}\right)[R_1^2 - r^2] + v_i \Rightarrow \text{para } 0 \le r \le R_1$$

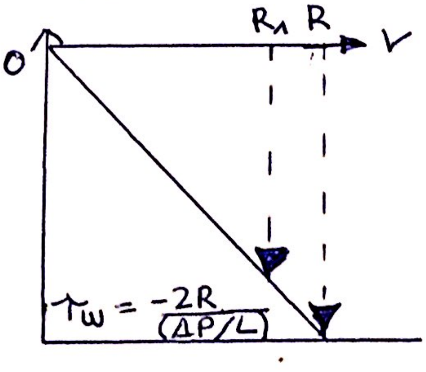
$$v_z^{(2)} = \frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)[R^2 - r^2] \quad \text{en } R_1 \le r \le R \text{ (shell)}$$

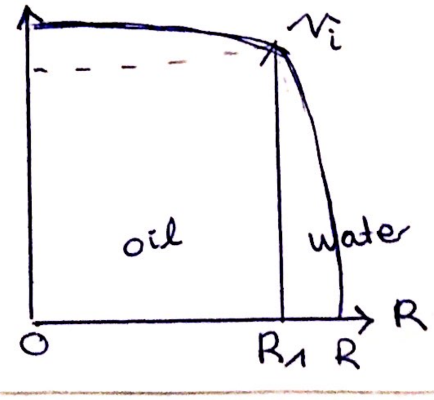
La velocidad en la interfase es:

$$v_z^{(2)}(r = R_1) = v_z^{(1)}(r = R_1) = v_i = \frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)[R^2 - R_1^2]$$

Esfuerzo cortante (cizallamiento) es negativo y aumenta linealmente a través del tubo (independiente de μ o si es flujo laminar o turbulento).

$$\tau_{rz}(r) = \frac{r}{2}\frac{\partial p}{\partial z} = -\frac{r}{2}\left(\frac{\Delta P}{L}\right)$$





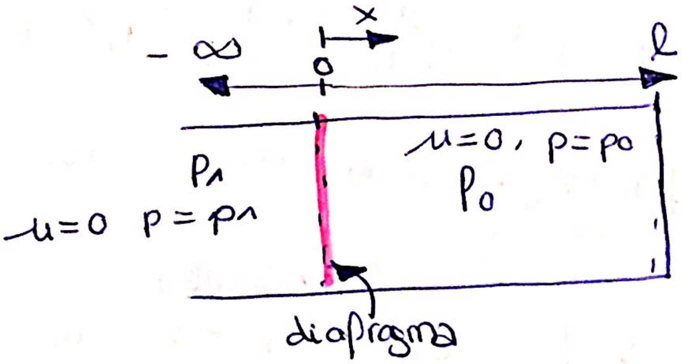
(campo de velocidad)
El vector velocidad es continuo pero cambia de pendiente con un factor $\frac{\mu_1}{\mu_2}$
en el limite de contorno $r = R_1$

f) $$\varphi_o = \int_0^{R_1} 2\pi r V_z^{(1)} dr = \frac{\pi}{2}\left(\frac{\Delta P}{L}\right)\int_0^{R_1} \frac{r}{\mu_1}\left[R_1^2 - r^2\right] + \frac{r}{\mu_2}\left[R^2 - R_1^2\right] dr = \frac{\pi}{4\mu_2}\left(\frac{\Delta P}{L}\right) R_1^2\left[R^2 - R_1^2\right] + \frac{\pi}{8\mu_1}\left(\frac{\Delta P}{L}\right) R_1^4 =$$

$$= \pi R_1^2 v_i + \underbrace{\frac{\pi}{8\mu_1}\left(\frac{\Delta P}{L}\right) R_1^4}_{\text{poiseuille}}$$

$$\varphi_w = \int_{R_1}^{R} 2\pi r V_z^{(2)} dr = \frac{\pi}{8\mu_2}\left(\frac{\Delta P}{L}\right)\left[R^2 - R_1^2\right]^2$$

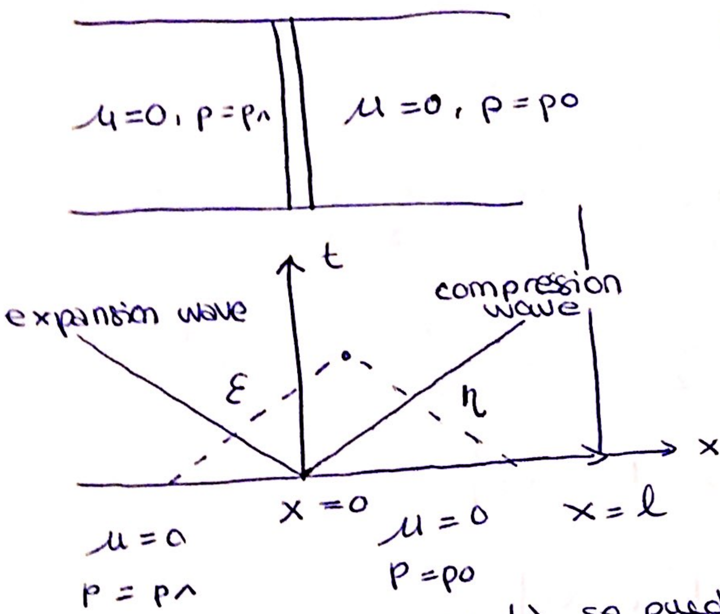
$\boxed{2}$



Point $P = (x, t)$:

$$\frac{u}{c} = \frac{1}{2\gamma}\left(\frac{p_1}{p_0} - 1\right)$$

$$\frac{p}{p_0} = \frac{1}{2}\left(\frac{p_1}{p_0} + 1\right)$$



$$\frac{p_1 - p_0}{p_0} \ll 1 \rightarrow p_1 \simeq p_0$$

La solución en el punto $p(x,t)$ se puede calcular siguiendo las líneas características

$\varepsilon = x - ct =$ constante

$$\frac{u}{c} + \frac{1}{\gamma}\frac{p}{p_0} = 0 + \frac{1}{\gamma}\frac{p_1}{p_0} \quad (1)$$

$\eta = x + ct =$ constante

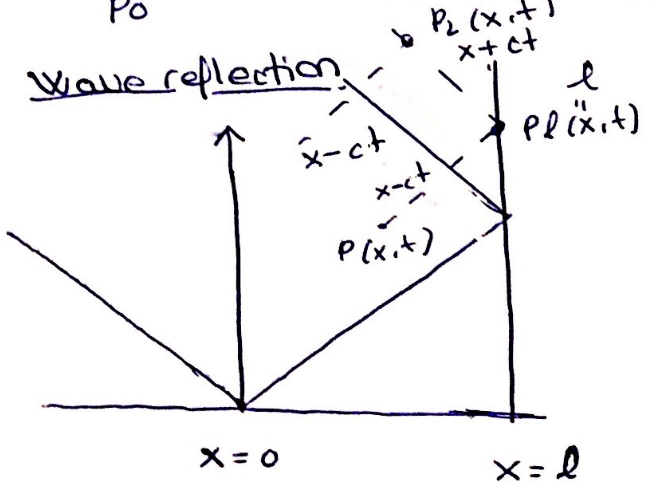
$$\frac{u}{c} - \frac{1}{\gamma}\frac{p}{p_0} = 0 - \frac{1}{\gamma}\cancel{\frac{p_0}{p_0}} \longrightarrow \boxed{\frac{u}{c} = \frac{1}{\gamma}\left(\frac{p}{p_0} - 1\right)} \quad (2)$$

$(2) \rightarrow (1)$

$$\cancel{\frac{1}{\gamma}}\left(\frac{p}{p_0} - 1\right) + \cancel{\frac{1}{\gamma}}\frac{p}{p_0} = \cancel{\frac{1}{\gamma}}\frac{p_1}{p_0}$$

$$\frac{2p}{p_0} - 1 = \frac{p_1}{p_0} \longrightarrow \boxed{p = \frac{p_0}{2}\left(\frac{p_1}{p_0} + 1\right) = \frac{p_0 + p_1}{2}}$$

<u>Wave reflection</u>



$x - ct = cte \rightarrow$

$$\frac{u}{c} + \frac{1}{\gamma}\frac{P_l}{P_0} = \frac{1}{\gamma}\overbrace{\left(\frac{P_0 + P_1}{2P_0} - 1\right)}^{\frac{P_1 - P_0}{2P_0}} +$$

$$+ \frac{1}{\gamma}\frac{P_0 + P_1}{2P_0}$$

$$\frac{P_l}{P_0} = \frac{1}{2P_0}(P_1 - P_0) + \frac{P_0 + P_1}{2P_0}$$

$$\boxed{P_l = \frac{P_1}{2} - \cancel{\frac{P_0}{2}} + \cancel{\frac{P_0}{2}} + \frac{P_1}{2} = P_1}$$

La solución en el punto $P_2(x,t)$ se puede calcular:

- $x-ct \Rightarrow \boxed{\dfrac{1}{2\gamma}\left(\dfrac{P_1-P_0}{P_0}\right)+\dfrac{1}{\gamma}\,\dfrac{\left(\dfrac{P_0+P_1}{2}\right)}{P_0} = \dfrac{u}{c}+\dfrac{1}{\gamma}\cdot\dfrac{P_2}{P_0}}$ (3)

- $x+ct$ (4) $\longrightarrow \left[\underbrace{\dfrac{u}{c}}_{=0} - \dfrac{1}{\gamma}\,\dfrac{\overset{P_1}{P_\ell}}{P_0} = \dfrac{u}{c} - \dfrac{1}{\gamma}\,\dfrac{P_2}{P_0} \rightarrow \dfrac{u}{c} = \dfrac{1}{\gamma}\left(\dfrac{P_2-P_1}{P_0}\right)\right.$

(4) → (3) $\quad \left(\dfrac{P_1-P_0}{P_0}\right)\dfrac{1}{2} + \dfrac{P_0+P_1}{2P_0} = \dfrac{P_2-P_1}{P_0} + \dfrac{P_2}{P_0}$

$$\frac{P_1}{P_0} = \frac{2P_2}{P_0} - \frac{P_1}{P_0} \longrightarrow \boxed{P_2 = P_1}$$

$$(4) \rightarrow \frac{u}{c} = 0 \qquad \boxed{u = 0}$$