AFM HomeWork - 2

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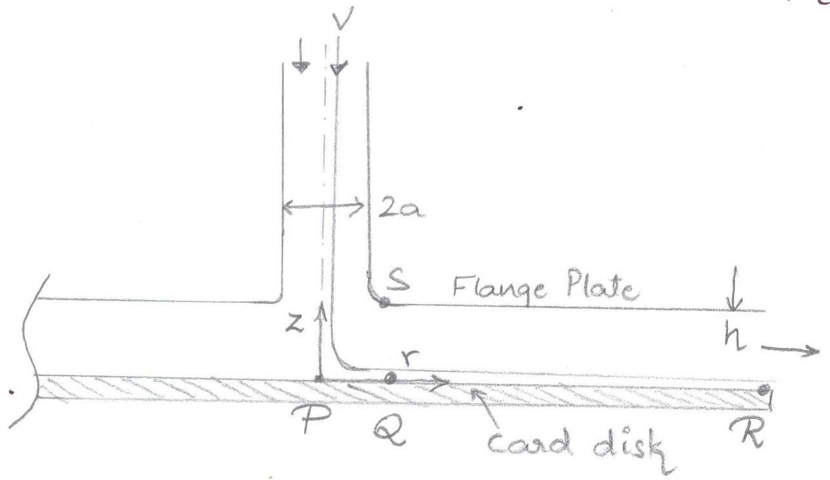
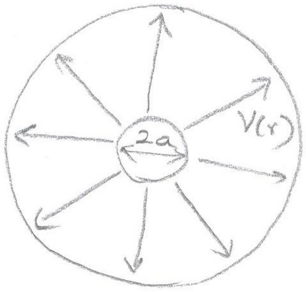
1)





$R >> a$
$h < a$ $\Rightarrow$ $p(r)$, $v(r) = f(r)$

a) $V_P = 0$, V_{PQ} accelerates
$V_r = C_1 \cdot r \quad r < a$

Show $V_r = C_2/r \quad r > a$

- Applying mass conservation, we get,

$V_0 \cdot A_0 = V_r A_r \rightarrow V_0 \cdot \pi a^2 = V_r \cdot 2\pi r . h \rightarrow V_r = V_0 \dfrac{a^2}{2rh} \quad r > a$

$V_r \Rightarrow \dfrac{C_2}{r} = V_0 \dfrac{a^2}{2rh} \rightarrow \boxed{C_2 = V_0 \dfrac{a^2}{2h}}$

$\boxed{r = a}$ $\quad V_{rPQ} = V_{rQR}$

$V_r \Rightarrow C_1 \cdot r = \dfrac{C_2}{r} \Rightarrow C_1 = \dfrac{C_2}{r^2} = \dfrac{V_0 a^2}{2h} \cdot \dfrac{1}{r^2} \overset{r=a}{=} \dfrac{a^2}{r^2} \cdot \dfrac{V_0}{2h}$

$\boxed{C_1 = \dfrac{V_0}{2h}}$

$V_P = 0 \qquad V_Q = \dfrac{V_0 a}{2h} \qquad V_R = V_0 \dfrac{a^2}{2hR}$

b) Evaluating the guage pressure field along stream line PQR

Applying Bernoulli's

$$\int_r^2 \frac{\partial v}{\partial t} ds + \left(\frac{1}{2}V_2^2 + \frac{P_2}{\rho} - F_2\right) - \left(\frac{1}{2}V_0^2 + \frac{P_0}{\rho} - F_0\right) = 0$$

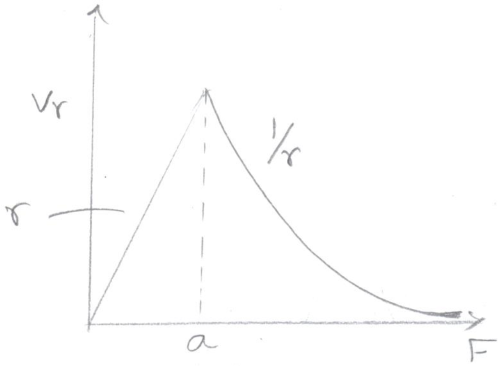
$$P_T = P_0 + \frac{1}{2}\rho V^2 = P(r) + \frac{1}{2}\rho\left[V_r(r)\right]^2$$

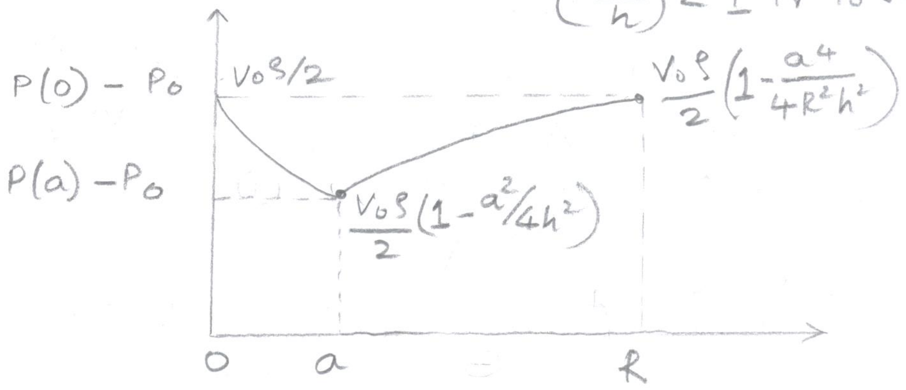
by substituting for V_r in each region $(r<a, r>a)$, we find,

$$P(r) - P_0 = \begin{cases} \frac{1}{2}\rho V^2\left[1 - r^2/4h^2\right] & r \le a \\ \frac{1}{2}\rho V^2\left[1 - a^4/4h^2r^2\right] & r \ge a \end{cases}$$

$\left(\frac{a}{h}\right)^2 > 1 \quad P_r - P_0 > 0$

$\left(\frac{a}{h}\right)^2 < 1 \quad P_r - P_0 < 0$





$p(r) = 0$

$r = 2h$

c) ~~Evalg~~ Force on Disk :

Integrate the answer for $P(r)$ over the entire disk to show that the total pressure acting on disks:

$$F_P = \frac{1}{2}\pi\rho V_0^2\left[R^2 - \frac{a^4}{8h^2}\right] + \pi\rho V_0^2\left(\frac{a^4}{4h^2}\right)\ln\left(\frac{a}{R}\right)$$

- $r \le a : P(r) - P_0 = \frac{\rho V_0^2}{2}\left[1 - \frac{r^2}{4h^2}\right]$
- $r \ge a : P(r) - P_0 = \frac{\rho V_0^2}{2}\left[1 - \frac{a^4}{4r^2h^2}\right]$

$$F_p = \int_0^a 2\pi r \frac{\rho V_0^2}{2}\left(1-\frac{r^2}{4h^2}\right)dr + \int_a^R 2\pi r \frac{\rho V_0^2}{2}\left(1-\frac{a^4}{4r^2h^2}\right)dr$$

$$= \int_0^a \frac{2\pi r \rho V_0^2}{2}\,dr - \int_0^a \frac{\pi r^3 \rho V_0^2}{4h^2} + \int_a^R \pi r \rho V_0^2\,dr - \int_a^R \frac{\pi r \rho V_0^2 a^4}{4r^2h^2}\,dr$$

$$= \pi \rho V_0^2 \int_0^R r\,dr - \frac{\pi \rho V_0^2}{4h^2}\int_0^a r^3\,dr - \frac{\pi \rho V_0^2 a^4}{4h^2}\int_a^R \frac{1}{r}\,dr$$

$$= \pi \rho V_0^2 \left[\frac{r^2}{2}\right]_0^R - \frac{\pi \rho V_0^2}{4h^2}\left[\frac{r^4}{4}\right]_0^a - \frac{\pi \rho V_0^2 a^4}{4h^2}\left[\ln(r)\right]_a^R$$

$$= \frac{\pi \rho V_0^2 R^2}{2} - \frac{\pi \rho V_0^2}{4h^2}\frac{a^4}{4} - \frac{\pi \rho V_0^2 a^4}{4h^2}\ln\left(\frac{R}{a}\right)$$

$$= \frac{1}{2}\pi \rho V_0^2\left(R^2 - \frac{a^4}{8h^2}\right) + \pi \rho V_0^2\left(\frac{a^4}{4h^2}\right)\ln\left(\frac{a}{R}\right)$$

$$F_p = \frac{1}{2}\pi \rho V_0^2\left[R^2 - \frac{a^4}{8h^2}\right] + \pi \rho V_0^2\left(\frac{a^4}{4h^2}\right)\ln\left(\frac{a}{R}\right)$$

The Given conditions are;

$F_p = -W$, $\rho_{air} = 1.2\ kg/m^2$, $a = 10^{-2} m$,

$R = 5e^{-2} m$, $h = 0.1 e^{-2} m$,

$W = mg = (10e^{-3})(9.8) = 9.8e^{-3}\ N$.

∴ Substituting the above values in the equation for F_p, we get,

$$-W = \frac{1}{2}\pi \rho V_0^2\left[R^2 - \frac{a^4}{8h^2}\right] + \pi \rho V_0^2\left(\frac{a^4}{4h^2}\right)\ln\left(\frac{a}{R}\right)$$

∴ $\boxed{V = 2.78\ m/s}$

d) Unsteady flow : The Bernoulli equation applied between points P and R gives,

$$\int_P^R \rho \frac{\partial V_r(r,t)}{\partial t}\, ds + \left[p(R) + \frac{1}{2}\rho V_R^2\right] - \left[P_{stag} + 0 + 0\right] = 0$$

The thing that vary's here is $h(t)$.
Useing conservation of mass with cylindrical control volume of radius r & height $h(t)$.
The expression for change in vel with time is,

$$\frac{dM_{sys}}{dt} = 0 = \frac{dM_{cv}}{dt} + \int_A \rho(v - v_c) \cdot n\, dA ,$$

In separate region is given by,

$$0 = \frac{d}{dt}(\pi r^2 h) - \pi r^2 V + 2\pi r h v_r \quad \text{for } r \leq a .$$

and,

$$0 = \frac{d}{dt}(\pi r^2 h) - \pi a^2 V + 2\pi r h V_r \quad \text{for } r \geq a .$$

$$V_r = -\frac{r}{2h}\frac{dh}{dt} + \frac{a^2 v}{2hr} \qquad \text{for } r > a$$

$$= C_1\left(\frac{dh}{dt}, h\right) r \qquad \text{for } r < a \quad \text{we get}$$

$$C_1\left(\frac{dh}{dt}, h\right) r \Big|_{r=a} = -a\frac{dh}{dt} + \frac{av}{2h}$$

$$C_1 = \frac{v}{2h} - \frac{dh}{2h\, dt}$$

$$V_r = \left(\frac{v - h}{2h}\right) r \qquad \text{for } r < a$$

$$V_r = \frac{Va^2}{2hr} - \frac{r}{2h}\frac{dh}{dt} \qquad \text{for } r \geq a$$

Applying bernoulli's, $r=0$ & $r=R$,

$$\int_0^R \frac{\partial V}{\partial t}\,ds + P_R + \frac{1}{2}\rho V_R^2 - P_2 - \cancelto{0}{\frac{1}{2}\rho V_2^2} = 0$$

$$P_0 = P_R$$

$$P_2 = P_1 + \frac{1}{2}\rho V^2 \quad \ldots\ldots \quad [P_1 = P_v]$$

$$= P_0 + \frac{1}{2}\rho V^2$$

$$\therefore\ P_R - P_2 = -\frac{1}{2}\rho V^2 \quad \ldots \quad P_0 = P_R.$$

$$V_r = \frac{V_0^2}{2Rh} - \frac{R}{2h}\left(\frac{dh}{dt}\right)$$

$$\int_0^a \frac{\partial v}{\partial t}\,ds + \int_0^R \frac{\partial V}{\partial t}\,ds = \frac{1}{2}\rho V^2 + \frac{1}{2}\rho\left[\frac{Va^2}{2Rh} - \frac{R}{2h}\frac{dh}{dt}\right]^2 = 0$$

$$\int_0^a \rho\left[\frac{-Vr}{2h^2}\frac{dh}{dt} + \frac{r}{2h^2}\left(\frac{dh}{dt}\right)^2 - \frac{r}{2h}\frac{d^2h}{dt^2}\right]dr + \int_a^R \rho\left[-\frac{Va^2}{2rh^2}\frac{dh}{dt} + \frac{r}{2h^2}\left(\frac{dh}{dt}\right)^2\right.$$

$$\left. - \frac{r}{2h}\frac{d^2h}{dt^2}\right]dr - \frac{1}{2}\rho V^2 + \frac{1}{2}\rho\left[\frac{Va^2}{2Rh} - \frac{R}{2h}\frac{dh}{dt}\right]^2 = 0$$

$$\cancel{\int_0^a \frac{d}{dt}\left(Vr - h(t)\right)}$$

(2)

Solution

Laplacian polar ~~coordinates~~ equation.

$$\nabla^2 a = \frac{\partial^2 a}{\partial r^2} + \frac{1}{r}\frac{\partial a}{\partial r} + \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2}$$

Given, $\psi(r,\theta) = f(r)\sin\theta$ with $f(r) = r^\alpha$.

$$\psi = r^\alpha \sin\theta$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \quad \text{— (1)}$$

$$\frac{\partial \psi}{\partial r} = \alpha r^{\alpha-1}\sin\theta, \qquad \frac{\partial^2 \psi}{\partial r^2} = (\alpha^2-\alpha)\, r^{\alpha-2}\sin\theta$$

$$\frac{\partial \psi}{\partial \theta} = r^\alpha \cos\theta, \qquad \frac{\partial^2 \psi}{\partial \theta^2} = -r^\alpha \sin\theta$$

$$\text{Eq (1)} \Rightarrow \nabla^2\psi = (\alpha^2-\alpha)\, r^{\alpha-2}\sin\theta + \frac{1}{r}\cdot\alpha r^{\alpha-1}\sin\theta + \frac{1}{r^2}(-r^\alpha \sin\theta)$$

$$= (\alpha^2-\alpha)\, r^{\alpha-2}\sin\theta + \alpha r^{\alpha-2}\sin\theta - r^{\alpha-2}\sin\theta$$

$$\nabla^2\psi = r^{\alpha-2}\sin\theta\left[\alpha^2 - \cancel{\alpha} + \cancel{\alpha} - 1\right)$$

$$\nabla^2\psi = r^{\alpha-2}\sin\theta\,(\alpha^2-1).$$

As we require plane flow to be irrotational,

$$\Delta\psi = 0$$

$$r^{\alpha-2}\sin\theta\,(\alpha^2-1) = 0 \Rightarrow \alpha^2 - 1 = 0$$

$$\text{or } \alpha = \pm 1.$$

$$a_1 \frac{\sin\theta}{r} + a_2 r \sin\theta$$

$$\psi = a_2 r \sin\theta + a_1 \frac{\sin\theta}{r} \quad \text{—— ②}$$

$$U_x = \sqrt{V_r^2 + V_\theta^2}$$

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$V_\theta = -\frac{\partial\psi}{\partial r}$$

$$\therefore V_r = \frac{1}{r}\left(\frac{a_1\cos\theta}{r} + a_2 r\cos\theta\right)$$

$$\boxed{V_r = \frac{a_1\cos\theta}{r^2} + a_2\cos\theta}$$

$$V_\theta = -\left(-\frac{a_1\sin\theta}{r^2} + a_2\sin\theta\right)$$

$$\boxed{V_\theta = \frac{a_1\sin\theta}{r^2} - a_2\sin\theta}$$

(a). Now as,

$$r \to \infty$$

$$V_r = a_2\cos\theta, \quad V_\theta = -a_2\sin\theta$$

$$V = U_x = \sqrt{V_r^2 + V_\theta^2}$$

$$= \sqrt{a_2^2\cos^2\theta + a_2^2\sin^2\theta}$$

$$\boxed{U_x = a_2}$$

Now consider,

at $r = R$, $\theta = \pi$, $V_r = 0$

$V_\theta = 0$

$$V_r = +\frac{a_1 \cos\pi}{R^2} + a_2 \cos\pi \qquad (\cos\pi = -1)$$

$$V_r = -\frac{a_1}{R^2} - U_x$$

$$0 = -\frac{a_1}{R^2} - U_x$$

$$\Rightarrow \boxed{a_1 = -U_x R^2}$$

Now substituting a_1 and a_2 in (21)

$$\boxed{\psi = -\frac{U_x R^2 \sin\theta}{r} + U_x r \sin\theta}$$

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = -\frac{U_x R^2}{r^2}\cos\theta + U_x \cos\theta.$$

$$V_r = U_x\left[1 - \frac{R^2}{r^2}\right]\cos\theta.$$

$$V_\theta = -\frac{\partial\psi}{\partial r} = +U_x \frac{R^2}{r^2}\sin\theta - U_x \sin\theta.$$

$$V_\theta = -U_x \sin\theta\left[1 + \frac{R^2}{r^2}\right]$$

At $r = R$,

$V_r = 0, \quad V_\theta = -2U_x \sin\theta.$

Hence, the appropriate boundary condition and expression in terms of stream function.

At $r = R, \theta = \pi$

$V_r = 0, \; V_\theta = 0$

$r \to \infty \quad \sqrt{\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)^2 + \left(-\frac{\partial\psi}{\partial r}\right)^2} = U_x.$

$\psi = f(r)$ at $r = R$.

(c) Velocity field.

$V = \sqrt{V_r^2 + V_\theta^2}$

$V = \sqrt{0 + (2U_x \sin\theta)^2}$

$V = \sqrt{4U_x^2 \sin^2\theta}, \quad \boxed{V = 2U_x \sin\theta}.$

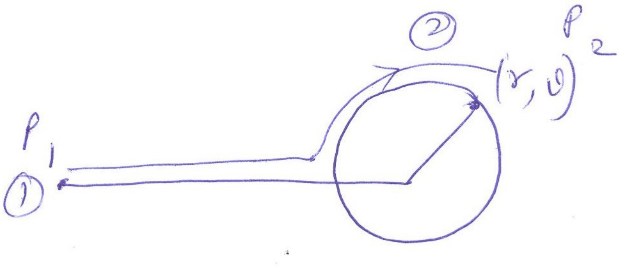
At $r = R$

$V_r = 0$

$V_\theta = -2U_x \sin\theta.$

(d).

Applying Bernoulli's Theorem between ① and ②.



$P_1 + \frac{1}{2}\rho V_x^2 = P_2 + \frac{1}{2}\rho(2U_x \sin\theta)^2$

$\Rightarrow P_1 + \frac{1}{2}\rho U^2 = P_2 + \frac{(4U^2 \sin^2\theta)}{2}\rho$

$\Rightarrow P_1 - P_2 = U^2\rho\left[2\sin^2\theta - \frac{1}{2}\right).$

$\Rightarrow P_1 - P_2 = U^2\rho\left[2\frac{(1-\cos 2\theta)}{2} - \frac{1}{2}\right]$

Pressure field, $P = P_2 - P_1$

$$\boxed{P = U^2 \rho \left(\cos 2\theta - \frac{1}{2}\right)}$$

(e). Net force acting on the cylinder.

$$F_H = \int_0^{2\pi} PR\,d\theta \cos\theta$$

$$F_V = \int_0^{2\pi} PR\,d\theta \sin\theta$$

$$F_X = \int_0^{2\pi} U^2 \rho \left(\cos 2\theta - \frac{1}{2}\right) R\,d\theta \cos\theta$$

$$= U^2 \rho R \left[\underbrace{\int_0^{2\pi} \cos 2\theta \cos\theta\,d\theta}_{\text{integrate by parts}} - \frac{1}{2}\int_0^{2\pi} \cos\theta\,d\theta \right]$$

$$\int_0^{2\pi} \cos 2\theta \cos\theta\,d\theta = \cos\theta \int_0^{2\pi} \cos 2\theta\,d\theta - \int_0^{2\pi} \frac{\sin\theta \sin 2\theta}{2}\,d\theta$$

$$= \cos\theta \int_0^{2\pi} \cos 2\theta\,d\theta - \left[\sin\theta \int_0^{2\pi} \sin 2\theta\,d\theta - \int_0^{2\pi} \frac{\cos 2\theta \cos\theta\,d\theta}{2} \right]$$

Let $\int_0^{2\pi} \cos\theta \cos 2\theta\,d\theta = P$

$$= \cos\theta \int_0^{2\pi} \cos 2\theta\,d\theta - \left[\sin\theta \int_0^{2\pi} \sin 2\theta\,d\theta - \frac{P}{2} \right]$$

$$\frac{P}{2} = \cos\theta \int_0^{2\pi} \cos 2\theta \, d\theta - \sin\theta \int_0^{2\pi} \sin 2\theta \, d\theta$$

$$\frac{P}{2} = \left[\cos\theta \cancel{\frac{\sin 2\theta}{2}}^{0}\right]_0^{2\pi} - \left[\sin\theta \cancel{\frac{\cos 2\theta}{2}}^{0}\right]_0^{2\pi}$$

$$P = 0$$

$$\therefore F_H = U^2 \rho R \left[0 - \frac{1}{2}[\sin\theta]\right]_0^{2\pi}$$

$$\boxed{F_H = 0}$$

$$F_V = \int_0^{2\pi} \rho R \sin\theta \, d\theta = \int_{-\pi}^{\pi} U^2 \rho R \left[\cos 2\theta - \frac{1}{2}\right] \sin\theta \, d\theta$$

$$= U^2 \rho R \times \int_{-\pi}^{0} \left(\cos(-2\theta) - \frac{1}{2}\right) \sin(-\theta) \, d\theta + \int_0^{\pi} \left(\cos 2\theta - \frac{1}{2}\right) \sin\theta \, d\theta$$

$$= U^2 \rho R \left[\int_0^{\pi} \left(\cos 2\theta - \frac{1}{2}\right)(-\sin\theta) \, d\theta + \int_0^{\pi} \left(\cos 2\theta - \frac{1}{2}\right) \sin\theta \, d\theta \right]$$

$$= U^2 \rho R \left[- \int_0^{\pi} \left(\cos\theta - \frac{1}{2}\right) \sin\theta \, d\theta + \int_0^{\pi} \left(\cos 2\theta - \frac{1}{2}\right) \sin\theta \, d\theta \right]$$

$$F_V = 0$$

$\therefore$ Net force acting on the cylinder,

$$F_x = \sqrt{F_H^2 + F_V^2} = 0.$$

$$\therefore \boxed{F_{net} = 0}$$

Comments :- This result makes sense according to D'Alembert's Paradox which states that drag force ($F = 0$) on a body moving with constant velocity relative to fluid.