Master in Numerical Methods in Engineering – Couple Problems Course 2016-2017

Lab and Theory Homework

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1 Introduction

The present document serves as evidence of the completion of the mandatory laboratory and theoretical homework covering transmission conditions, domain decomposition methods for both homogeneous and heterogeneous problems, monolithic and partitioned schemes in time, operator splitting techniques, fractional step methods, ALE formulations and Fluid-Structure interaction problems.

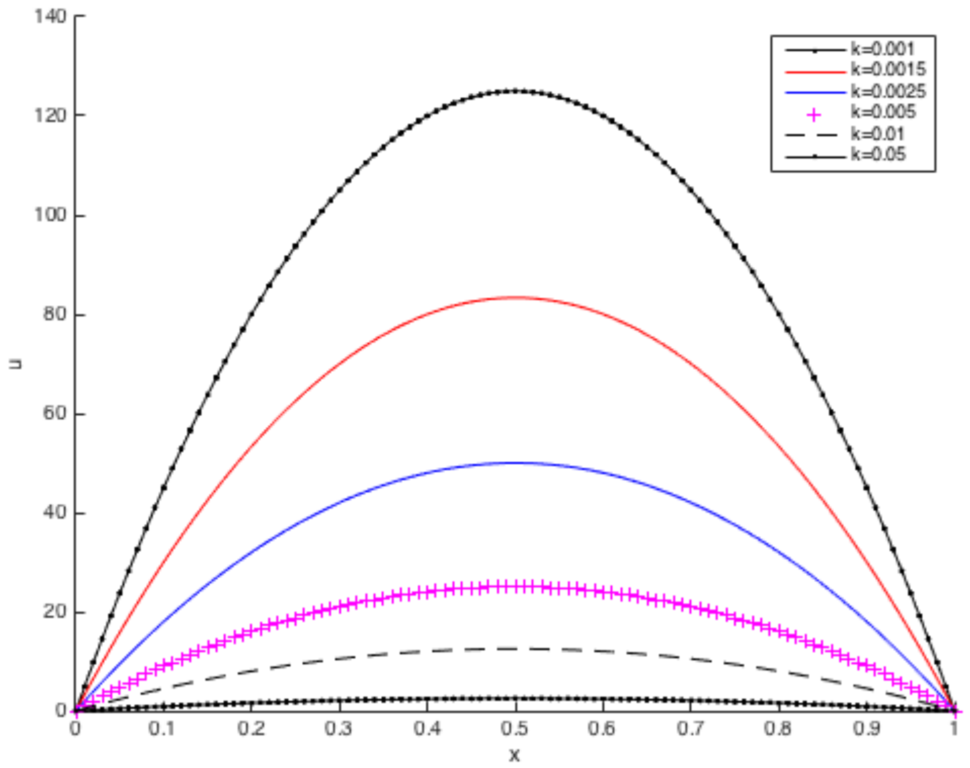
The results of the lab homework are summarized in Section 1, while the answers to the theoretical homework are presented in the Appendices Section of the present document.

2 Lab

2.1 Task 1: Heat Transfer problem

The influence of the value of kappa on the solution of the heat transfer have been computed and presented in Figure 2.1. As expected we observe that if we increase the diffusion effects (larger kappa-value), the solution decreases which is the expected behavior.

Figure 2.1. Effect of varying kappa-value



The influence of the value of the source term on the solution of the heat transfer have been computed and presented in Figure 2.2. As expected we observe that if we increase the magnitude of the source, the solution increases accordingly which is the expected behavior as they are proportionally to each other.

The influence of the number of elements in which the domain is discretized on the solution of the heat transfer have been computed and presented in Figure 2.3. As expected we observe that for finer mesh (larger number of element) the overall solution is better approximated. Nevertheless, the approximation at the nodes are equally approximated independently of the number of elements as we can observe at Figure 2.3.

Figure 2.2. Effect of varying the source term

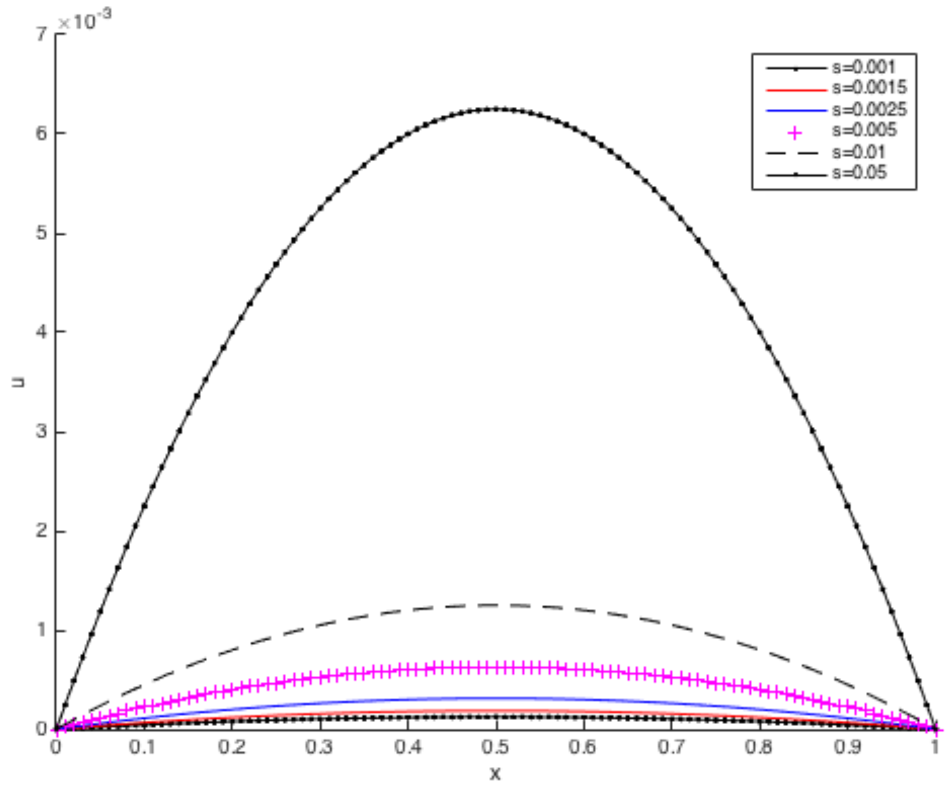
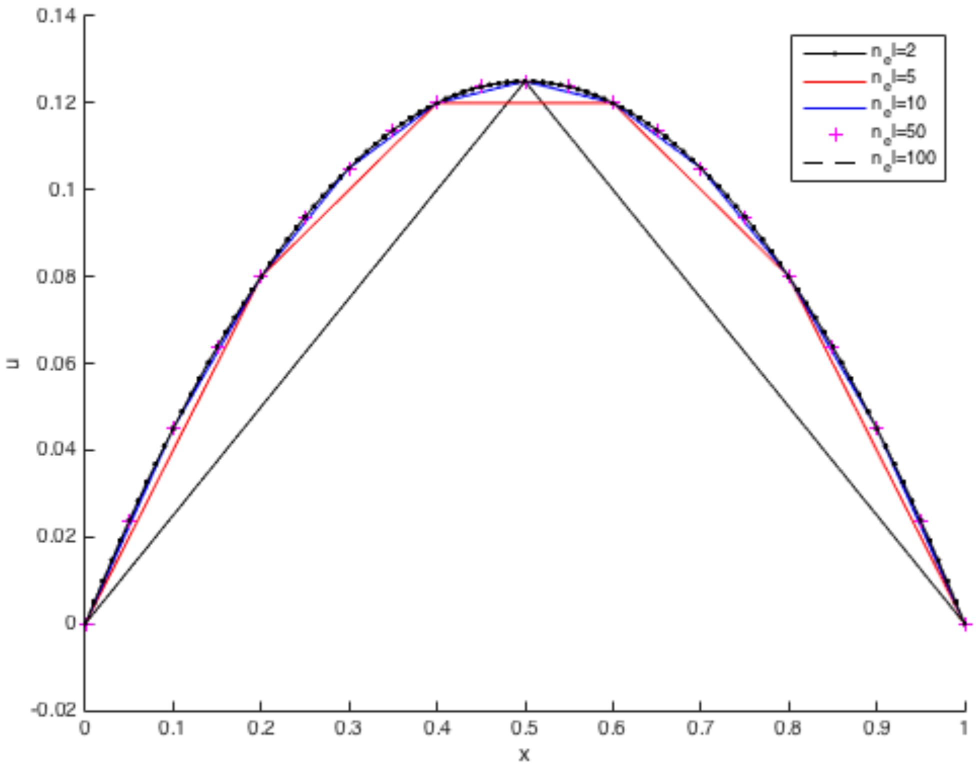


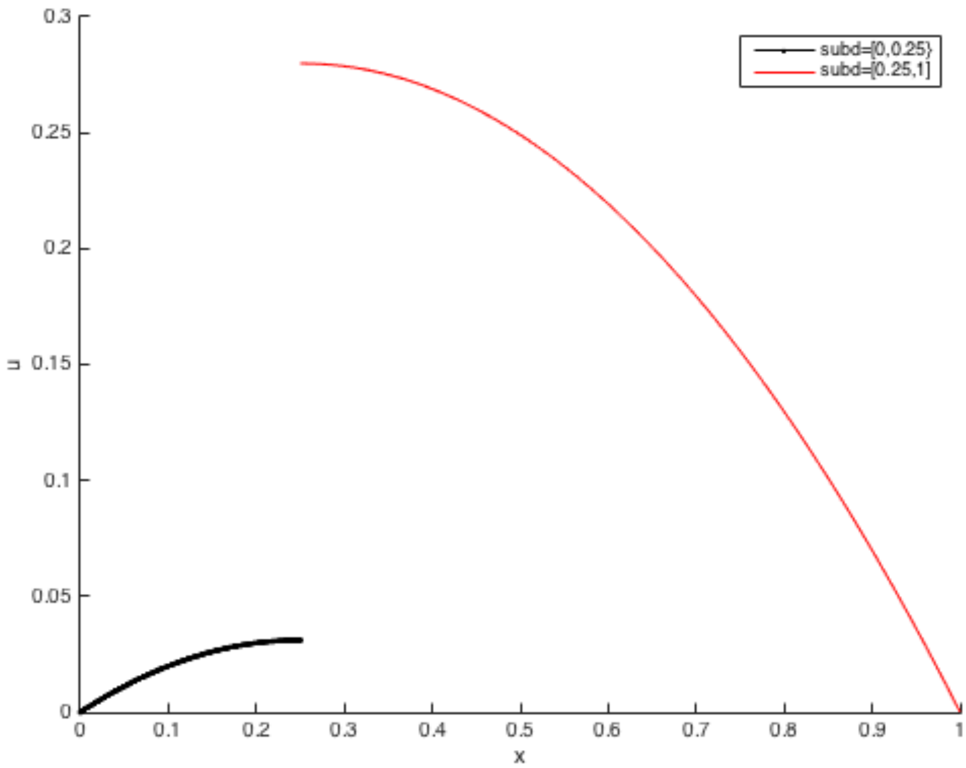
Figure 2.3. Effect of varying the number of elements



2.2 Task 2: Solve a Heat Tranfer problem split into two subdomains

The heat transfer problem is now split into two subdomains Ω1=[0,0.25] and Ω2=[0.25,1] with kappa=1 and source=1, leaving the interface node free of prescribed boundary conditions. The latter causes mismatching approximated values at the shared interface node, as can be observed in Figure 2.4 below. The reason is simple right end-node of subdomain Ω1 does not talk to the left end-node of subdomain Ω1 and therefore they are unsynchronized.

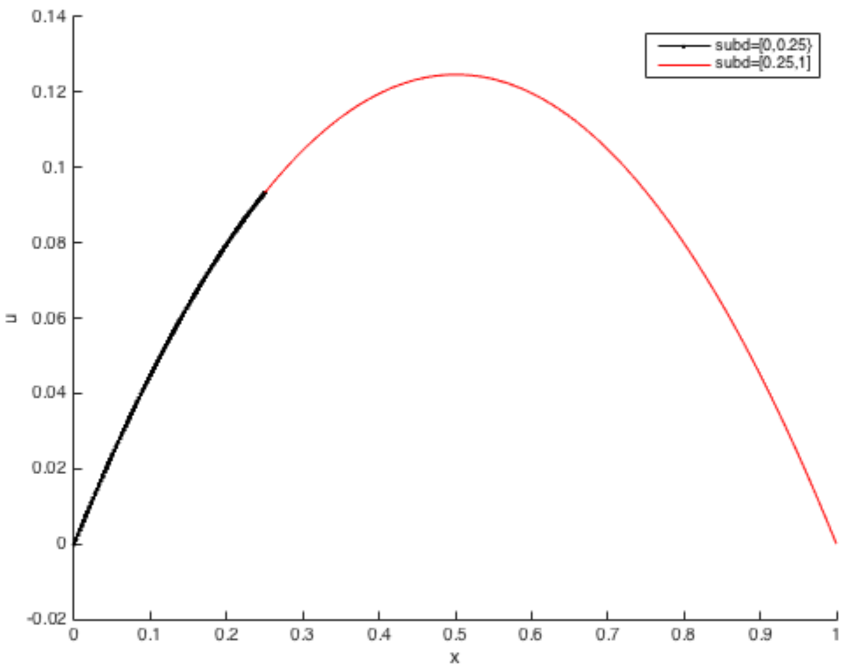
Figure 2.4. Effect of free conditions at the shared interface of the two subdomains



2.3 Task 3: Monolithicly solve the Heat Transfer problem split into two subdomains

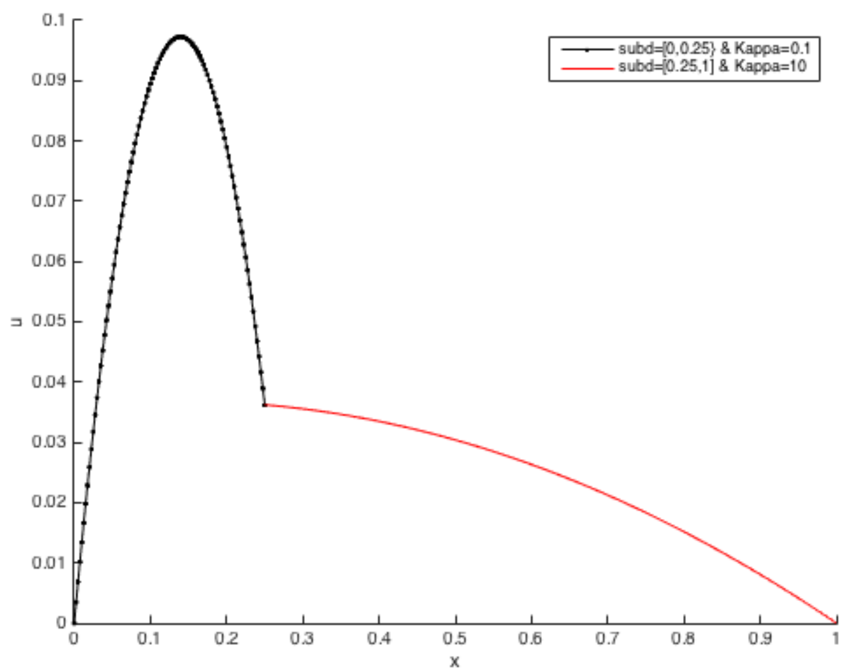
If we were to solve the problem presented in section 2.2 with a monolithic approach, we will manage to obtain a matching solution at the shared interface (see Figure 2.5). The reason is that thanks to the monolithic approach the two sub-domains, while still having free boundary conditions at the shared interface, they now share the same equations and as the mesh nodes coincide at the interface it is no longer required to integrate the boundary terms.

Figure 2.5. two subdomains with free shared interface boundary conditions solved using a monolithic scheme.



If we were to introduce a different value of kappa at each subdomain, the solution at the shared interface node would still be a matching solution for both subdomains when a monolithic approach is implemented. The differences are that instead of having a single problem with two subdomains, now we have what looks like two different problems with a shared interface (see Figure 2.6). The larger the difference between kappa1 and kappa2 the weirdest the shape of the solution curve will be.

Figure 2.6. two subdomains with free shared interface boundary conditions solved using a monolithic scheme- with different kappa values.



2.4 Task 4. Solve the Heat Transfer problem with a Dirichlet-Neumann iteration-by-subdomain scheme

The iterative scheme converges well for different starting neumann boundary conditions at the left subdomain, with the requisite that kappa shall be equal or greater than 1, as shown in Figures 2.7 through 2.10. The method converges faster for increasing values of kappa and does not converge at all for values below 1 (see Figure 2.11).

The limitations of this scheme are the difficulties to guarantee the convergence and stability of the solution.

Figure 2.7. Dirichlet-Neumann iteration-by-subdomain for q1=1 and k1=k2=1.

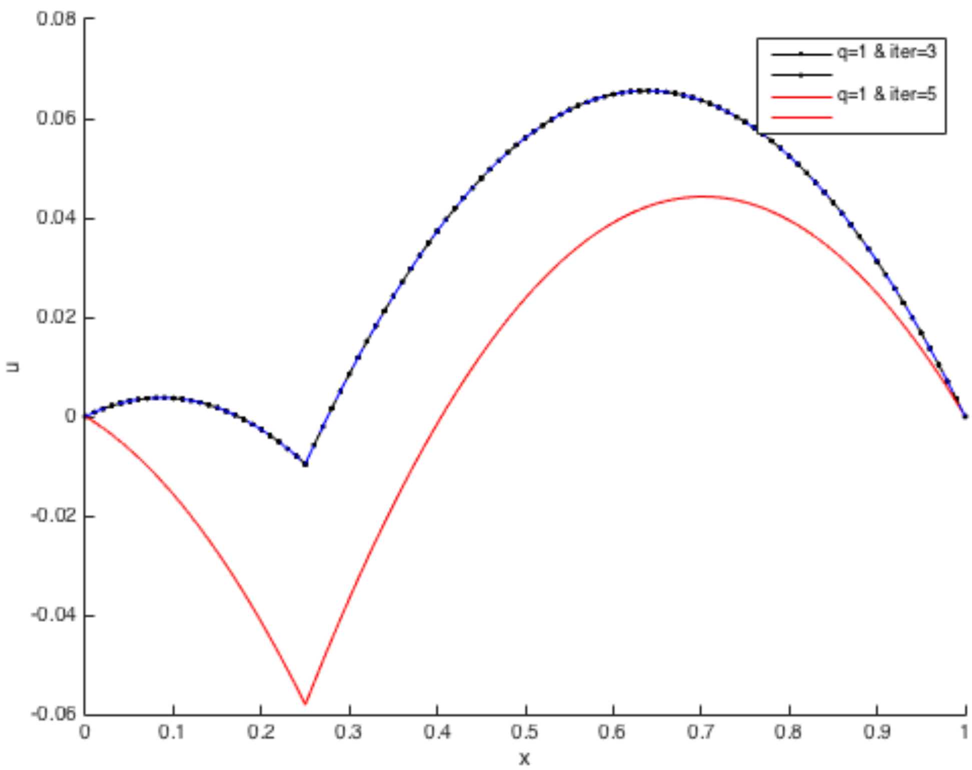


Figure 2.8. Dirichlet-Neumann iteration-by-subdomain for q1=100 and k1=k2=1.

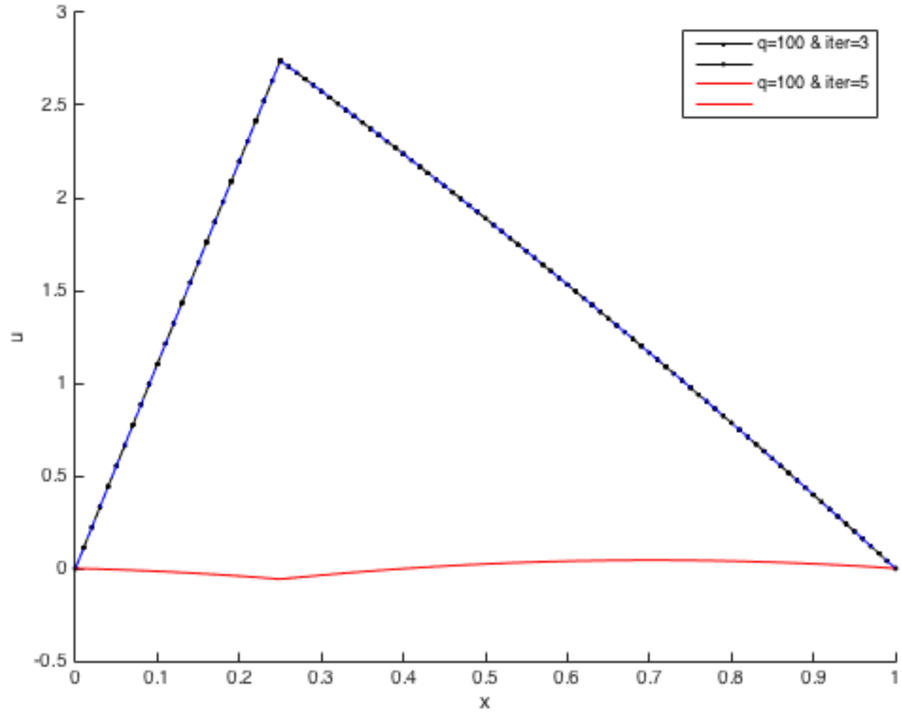


Figure 2.9. Dirichlet-Neumann iteration-by-subdomain for q1=1 with k1=100 and k2=1.

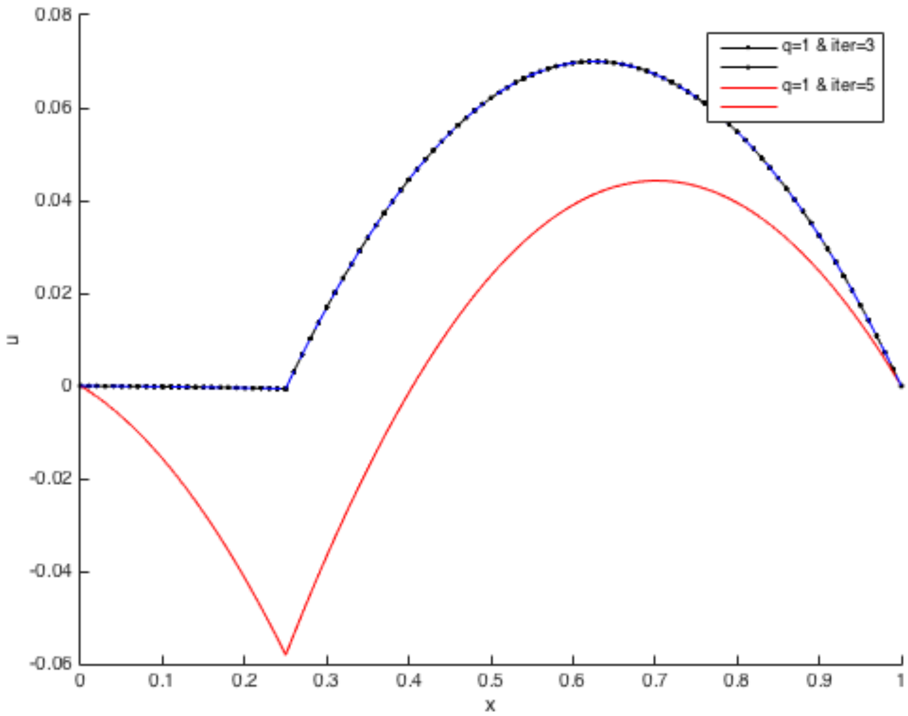


Figure 2.10. Dirichlet-Neumann iteration-by-subdomain for q1=100 with k1=100 and k2=1.

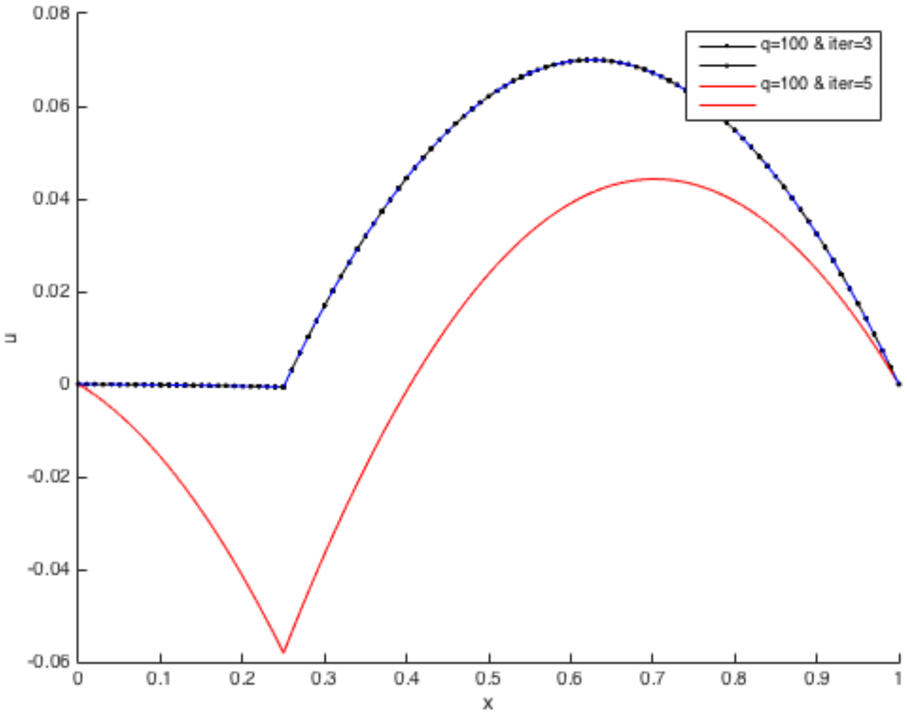
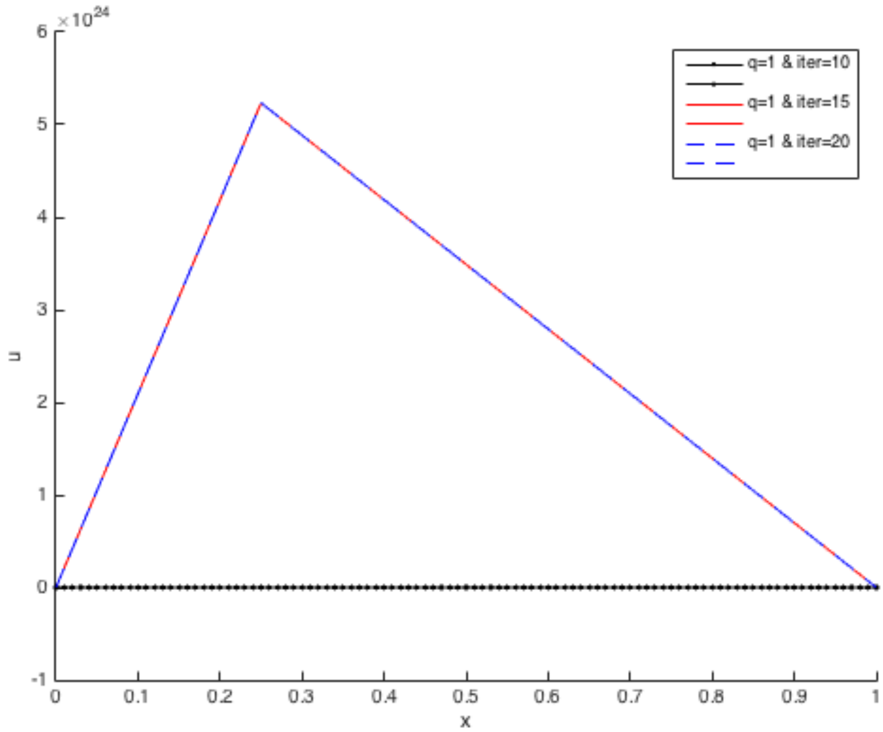


Figure 2.11. Dirichlet-Neumann iteration-by-subdomain for q1=0.01 with k1=100 and k2=1.



3 References

Codina R., and Bages J., Notes of Coupled Problems Course. Master of Numerical Methods in Engineering. CIMNE 2017

4 Appendices

4.1 THEORETICAL HOMEWORK

COUPLED PROBLEMS

① TRANSMISSION CONDITIONS

[1] (a) given that both δv & v must have continuous second spatial derivatives $\frac{d^2 \delta v}{dx^2}$ & $\frac{d^2 v}{dx^2}$ & these must be square integrable to comply with the regularity requirements, their space of functions are as follows:

$$\mathcal{S} = \{ v \mid v \in H^2(\Omega), \ v(0) = v(L) = 0 \}$$

$$\mathcal{V} = \{ \delta v \mid \delta v \in H^2(\Omega), \ \delta v(0) = \delta v(L) = 0 \}$$

[1] (b) the transmission conditions at P implied by regularity condition ~~must~~ are $[\![v]\!]_\Gamma = 0$, where Γ is P.

the actual t. conditions for ~~these~~ clamped Bernoulli-Euler problem are:

$$EI \left(\int_{\Omega_1} \delta v \frac{d^4 v}{dx^4} + \int_{\Omega_2} \delta v \frac{d^4 v}{dx^4} \right) =$$

$$= EI \Bigg[- \int_{\Omega_1} \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} + \int_\Gamma \frac{d \delta v}{dx} n_1 \cdot \frac{d^2 v}{dx^2} - \int_\Gamma \delta v \, n_1 \cdot \frac{d^3 v}{dx^3}$$

$$- \int_{\Omega_2} \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} + \int_\Gamma \frac{d \delta v}{dx} n_2 \cdot \frac{d^2 v}{dx^2} - \int_\Gamma \delta v \, n_2 \cdot \frac{d^3 v}{dx^3} \Bigg]$$

[1] (c) if the integral is additive the $\sum \int_{\Omega_n}$ integrals at both ~~...~~ RHS & LHS are equal and the system is reduced to the sum of boundary terms ~~equ~~ being null ~~[illegible]~~

$$[\![K \frac{\partial u}{\partial n}]\!]_\Gamma = 0$$

$$\hookrightarrow \left(\int_\Gamma \frac{d \delta v}{dx} \frac{d^2 v}{dn_1^2} - \int_\Gamma \delta v \frac{d^3 v}{dn_1^3} + \int_\Gamma \frac{d \delta v}{dx} \frac{d^2 v}{dn_2^2} - \int_\Gamma \delta v \frac{d^3 v}{dn_2^3} \right) = 0$$

① cont'd.

[2] (a) find $u \in H^2(\Omega)$ such that

$$\int_\Omega \delta v \left(\nu \nabla \times \nabla \times u\right) = \int_\Omega \delta v\, f$$ Knowing that

$\nabla \times \nabla \times u = \nabla(\nabla \cdot u) - \nabla^2 u$ the above integral system can be presented as:

$$\int_\Omega \delta v \left(\nu \nabla(\nabla \cdot u)\right) - \int_\Omega \delta v \left(\nu \nabla^2 u\right) = \int_\Omega \delta v\, f$$

$$\rightarrow -\int_\Omega \nabla \delta v\, \nu\, (\nabla \cdot u) + \int_\Gamma \delta v\, \nu\, (\nabla \cdot u) n - \int_\Omega \delta v\, \nu\, \nabla^2 u = \int_\Omega \delta v f$$

where $\nabla \cdot u = 0$ in Ω leads to

$$\int_\Gamma \delta v\, \nu\, (\nabla \cdot u) n - \int_\Omega \delta v\, \nu\, \nabla^2 u = \int_\Omega \delta v f$$

The space of functions of u must satisfy continuity of $\nabla^2 u$ therefore, $\underline{\mathcal{S} = \{u \mid u \in H^2(\Omega)\}}$ is space of functions of u.

[2] (b) $[\![u]\!]_\Gamma = 0$ the transmission condition is:

$$\int_{\Omega_1} \delta v \left(\nu \nabla \times \nabla \times u\right) + \int_{\Omega_2} \delta v \left(\nu \nabla \times \nabla \times u\right) =$$

$$= \int_\Gamma \delta v\, \nu\, (\nabla \cdot u) n_1 - \int_{\Omega_1} \delta v\, \nu\, \nabla^2 u + \int_\Gamma \delta v\, \nu\, (\nabla \cdot u) n_2 - \int_{\Omega_2} \delta v\, \nu\, \nabla^2 u$$

[2] (c) if the integral is additive $\int_\Omega = \int_{\Omega_1} + \int_{\Omega_2}$ & then we obtain that $[\![K \frac{\partial u}{\partial n}]\!]_\Gamma = 0$ leading to the following t. conditions:

$$\int_\Gamma \delta v\, \nu\, (\nabla \cdot u) n_1 + \int_\Gamma \delta v\, \nu\, (\nabla \cdot u) n_2 = 0$$

① cont'd

[3] (a)

the first equation $-2\mu\nabla\cdot(\varepsilon(u)) - \lambda\nabla(\nabla\cdot u) = \rho b$ where $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ leads to

$-\mu\nabla\cdot(\nabla u + \nabla^T u) - \lambda\nabla(\nabla\cdot u) = \rho b$ that leads to the variational:

$$\int_\Omega \nabla\delta v\,\mu\,\nabla u - \int_\Gamma \delta v\,\mu\,\nabla u\cdot n - \int_\Omega \delta v\,\mu\,\nabla\cdot\nabla^T u - \int_\Omega \nabla\delta v\,\lambda\,(\nabla\cdot u) = \int_\Omega \delta u\,\rho b$$

as $u=0$ on Γ $\quad 0$

$$\int_\Omega \nabla\delta v\,\mu\nabla u - \int_\Omega \nabla\delta v\,(\lambda+\mu)(\nabla\cdot u) = \int_\Omega \delta v\,\rho b.$$

the second equation $-\mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u) = \rho b$ which yields. $-\mu\nabla\cdot(\nabla u) - (\lambda+\mu)\nabla(\nabla\cdot u) = \rho b$ which leads to the same variational as before.

the third equation $(\mu\nabla\times(\nabla\times u) - (\lambda+2\mu)\nabla(\nabla\cdot u) = \rho b$ where $\mu\nabla\times(\nabla\times u) = \mu(\nabla(\nabla\cdot u) - \nabla^2 u)$ can be translated as: $-\mu\nabla^2 u - (\lambda+\mu)\nabla(\nabla\cdot u) = \rho b$ leading to the same variational as before.

[3] (b) $[\![u]\!]_\Gamma = 0$ & $[\![K\frac{\partial u}{\partial n}]\!]_\Gamma = 0$

where the second will look like:

$$\int_\Gamma \delta v\,\mu\,\nabla u\cdot n_1 + \int_\Gamma \delta v\,\mu\,\nabla u\cdot n_2 + \int_\Gamma \delta v\,(\lambda+\mu)(\nabla\cdot u)\cdot n_1 + \ldots$$

$$\ldots + \int_\Gamma \delta v\,(\lambda+\mu)(\nabla\cdot u)\cdot n_2 = 0$$

② Domain Decomposition Methods

[1] (a) Schwarz additive DD method

$$\mathcal{L}\, v_1^{(k)} = f \quad \text{in } \Omega_1 \qquad\qquad \mathcal{L}\, v_2^{(k)} = f \quad \text{in } \Omega_2$$

$$v_1^{(k)} = 0 \quad \text{on } \Gamma_1 \qquad\qquad v_2^{(k)} = 0 \quad \text{on } \Gamma_2$$

$$v_1^{(k)} = v_2^{(k-1)} \quad \text{on } \Gamma_{12} \qquad\qquad v_2^{(k)} = v_1^{(k-1)} \quad \text{on } \Gamma_{12}$$

where :

$$\mathcal{L}\, v_1 = \underline{f} \longrightarrow \begin{cases} \mathcal{L}\, v_1 = EI \int_0^{L_1} \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} \\ \underline{f} = \int_0^{L_1} \delta v\, f \end{cases}$$

$$\& \quad \mathcal{L}\, v_2 = \underline{f} \longrightarrow \begin{cases} \mathcal{L}\, v_2 = EI \int_{L_2}^{\cancel{L_2}} \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} \\ \underline{f} = \int_{L_2}^{L} \delta v\, f \end{cases}$$

[~~1~~] (b) the matrix version $AU = F$

$$\begin{bmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_\Gamma \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma \\ F_2 \end{bmatrix}$$

where: $A_{11} = \int_0^{L_1} \mathcal{L} v_1$; $A_{22} = \int_{L_2}^{L} \mathcal{L} v_2$; $A_{1\Gamma} = A_{\Gamma 1} = A_{2\Gamma} = A_{\Gamma 2} = 0$

$A_{\Gamma\Gamma}$: transmission conditions

$$F_1 = \int_0^{L_1} \delta v\, f = \int_0^{L_1} \underline{f} \quad ; \quad F_2 = \int_{L_2}^{L} \delta v\, f = \int_{L_2}^{L} \underline{f} \quad ; \quad F_\Gamma = \int_{L_2}^{L_1} \delta v\, f$$

where $\mathcal{L} v_1$ & $\mathcal{L} v_2$ are the ones stated in [1] (a)

② Cont'd

[2] (a) Iteration by sub-domain based on the Dirichlet-Neumann coupling.

$$\mathcal{L} u_1^{(k)} = f \quad \text{in } \Omega_1 \qquad\qquad \mathcal{L} u_2^{(k)} = f \quad \text{in } \Omega_2$$

$$u_1^{(k)} = \bar{u}\,|_{\Gamma_1} \quad \text{on } \Gamma_1 \qquad\qquad u_2^{(k)} = \bar{u}\,|_{\Gamma_2} \quad \text{on } \Gamma_2$$

$$\underbrace{k_1 \frac{\partial u_1^{(k)}}{\partial n} = k_2 \frac{\partial u_2^{(k-1)}}{\partial n} \quad \text{on } \Gamma_{12}}_{\text{Neumann BCs that}} \qquad\qquad u_2^{(k)} = u_1^{(k-1)} \quad \text{on } \Gamma_{12}$$

for the Maxwell problem will look like this:

$$\underline{\nu (\nabla \cdot u_1^{(k)}) \cdot n = \nu (\nabla \cdot u_2^{(k-1)}) \cdot n} \quad \text{however as } \nabla \cdot u = 0 \text{ in } \Omega$$

and $\Gamma_{12} \in \Omega \implies \underbrace{\nabla \cdot u_1^{(k)} = \nabla \cdot u_2^{(k-1)} = 0}_{\text{N.BCs.}}$

[2] (b) the Steklov-Poincaré operator of the problem

$\boxed{\mathcal{S}\varphi = \mathcal{G}}$ where $\mathcal{S}$: steklov-Poincaré operator

& $\varphi = \nu (\nabla \cdot u_1^{(k)}) \cdot n - \nu (\nabla \cdot u_2^{(k-1)}) \cdot n$

& $\mathcal{G} = \nu (\nabla \cdot u_1^{1}) n - \nu (\nabla \cdot u_2^{0}) \cdot n$

[2] (c) the matrix version of the (a) scheme is:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} U_1^{(k)} \\ U_2^{(k)} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2} U_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)} U_\Gamma^{(k-1)} \end{bmatrix}$$

$$A_{22} U_2^{(k)} = F_2 - A_{2\Gamma} U_\Gamma^{(k-1)}$$

where the terms are as per ② [1] (b).

② cont'd

[3] (a) iteration-by-subdomain based on the Dirichlet-Robin coupling:

$$\Omega_1 \begin{cases} \mathcal{L} v_1^{(k)} = f & \text{in } \Omega_1 \\ v_1^{(k)} = \bar{v}|_{\Gamma_1} & \text{on } \Gamma_1 \\ v_1^{(k)} = \bar{v}|_{\Gamma_1} & \text{on } \Gamma_{12} \end{cases} \qquad \Omega_2 \begin{cases} \mathcal{L} v_2^{(k)} = f & \text{in } \Omega_2 \\ v_2^{(k)} = \bar{v}|_{\Gamma_2} & \text{on } \Gamma_2 \\ K_1 \dfrac{\partial v_1^{(k)}}{\partial n} + \gamma_1 v_1^{(k)} = K_2 \dfrac{\partial v_2^{(k-1)}}{\partial n} + \gamma_i v_2^{(k-1)} & \text{on } \Gamma_{12} \end{cases}$$

[3] (b)

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} U_1^{(k)} \\ U_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2} U^{(k-1)} - A_{\Gamma\Gamma}^{(2)} U_\Gamma^{(k-1)} \end{bmatrix}$$

$$A_{22} U_2^{(k)} = F_2 - A_{2\Gamma} U_\Gamma^{k-1}$$

where:

$A_{\Gamma\Gamma}^{(1)} \cdot U_\Gamma^{(k-1)}$: Dirichlet BC

$A_{\Gamma\Gamma}^{(2)} \cdot U_\Gamma^{(k-1)}$: Robin BC

[3] (c) Schur (S) complement as discrete version of the Steklov-Poincaré operator

$$S = \left(A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - A_{\Gamma 2} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}\right)$$

[3] (d)

3 COUPLING OF HETEROGENEOUS PROBLEMS

[1] (a)

$$\begin{cases} \sigma_{xx} = \dfrac{E}{1-\nu^2}\left(\dfrac{\partial u}{\partial x} + \nu \dfrac{\partial u}{\partial y}\right) = 0 \\ \sigma_{yy} : \dfrac{E}{1-\nu^2}\left(\nu \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\right) = -\rho_{beam} \cdot g + \rho_{plate} \cdot g \left(\dfrac{y}{L}\right) \\ u(0,y) = u(x,-L) = u(L,y) = 0 \\ \sigma_{zz} = 0 \end{cases}$$

4 MONOLITHIC AND PARTITIONED SCHEMES IN TIME

[1] $\dfrac{\partial u}{\partial t} - k \dfrac{\partial^2 u}{\partial x^2} = f$ in $[0,1]$

$$u(x=0,t) = u(x=1,t) = u(x,t=0) = 0$$

where $k=1$, $f=1$ & $\delta t = 1$.

the week form looks like this: $\displaystyle\int_\Omega \delta v \frac{\partial u}{\partial t} + \int_\Omega \nabla \delta v \cdot k \nabla u = \int_\Omega \delta v f$

No Neumann BCs prescribed

and the BDF1 approximation : $\dfrac{\partial u^{n+1}}{\partial t} = \dfrac{(u^{n+1} - u^n)}{\delta t}$

for which the resulting matrix system is:

$$M \frac{(u^{n+1} - u^n)}{\delta t} + K u^{n+1} = F^{n+1} \quad , \text{ where:}$$

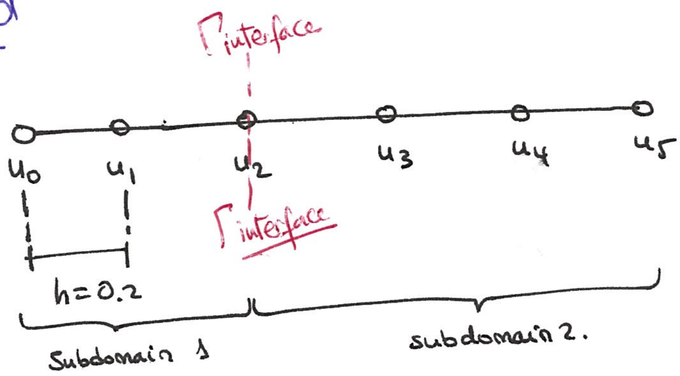
$$\begin{cases} M = \displaystyle\int_\Omega \delta v_i \, \delta v_j = \int_\Omega N_i N_j \\ K = \displaystyle\int_\Omega \nabla \delta v_i \cdot k \nabla \delta v_j = \int_\Omega \frac{dN_i}{dx} k \frac{dN_j}{dx} \\ F = \displaystyle\int_\Omega \delta v_i f = \int_\Omega N_i f \end{cases}$$

yielding the following relationship:

$$u^{n+1} = \frac{(F^{n+1} + M u^n)}{(M + K)} \quad \text{where } u^{n+1} = \begin{bmatrix} u(x=0,t) \\ u(x=1,t) \end{bmatrix}$$

4 Cont'd

[2]



→ Subdomain 1 eq: $(\delta v_1, \partial_t u_1^{n+1})_{\Omega_1} + (\nabla \delta v_1, \nabla u_1^{n+1})_{\Omega_1} + \dots$

$$\dots - \langle \delta v_1, n_1 \cdot \nabla u_1^{n+1} \rangle_{\Gamma_{int}} = \langle \delta v_1, f^{n+1} \rangle_{\Omega_1}$$

→ Subdomain 2 eq: $(\delta v_2, \partial_t u_2^{n+1})_{\Omega_2} + (\nabla \delta v_2, \nabla u_2^{n+1})_{\Omega_2} + \dots$

$$\dots - \langle \delta v_2, n_2 \cdot \nabla u_2^{n+1} \rangle_{\Gamma_{int}} = \langle \delta v_2, f^{n+1} \rangle_{\Omega_2}$$

As the discretization size is the same at both subdomains and their meshes match at the interface:

$$n_1 \cdot \nabla u_1^{n+1} + n_2 \cdot \nabla u_2^{n+1} = 0$$

and if we sum both equations we ~~obtain~~ ~~determin~~ discover that they are the same and therefore there is no need ~~for~~ for boundary integrals at the ~~boundary~~. interface.

4 Cont'd

[3] Algebraic form of the Dirichlet-to-Neumann operator, for the left subdomain.

the transmission conditions at the interface are:

$u_1 = u_2$; $K_1 \cdot \frac{\partial u_1}{\partial n} = K_2 \cdot \frac{\partial u_2}{\partial n}$ on Γ_{12} ; where $\Gamma_{12} = \Omega_1 \cap \Omega_2$

the Steklov-Poincare operator can be derived as follows:

$u_i = u_i^0 + \tilde{u}_i$ | $\mathcal{L} \tilde{u}_i = 0$ in Ω_i

$\mathcal{L} u_i^0 = f$ in Ω_i | $\tilde{u}_i = u|_{\Gamma_i}$ on Γ_i

$u_i^0 = 0$ on Γ_i | $\tilde{u}_i = \varphi$ on Γ_{12}

$u_i^0 = 0$ on Γ_{12} | such that $u_i = u_i^0 + \tilde{u}_i$ is $u|_{\Omega_i}$

where $i = 1, 2$.

ensuring that $K_1 \cdot \frac{\partial u_1}{\partial n} = K_2 \frac{\partial u_2}{\partial n} \Longleftrightarrow K_1 \cdot \frac{\partial \tilde{u}_i}{\partial n} - K_2 \cdot \frac{\partial \tilde{u}_2}{\partial n} = -K_1 \cdot \frac{\partial u_i^0}{\partial n} + K_2 \cdot \frac{\partial u_2^0}{\partial n}$

Define the SP operator $\mathcal{J}$ as:

$$\mathcal{J} : H^{1/2}(\Gamma_{12}) \longrightarrow H^{-1/2}(\Gamma_{12}) \quad \Big\{ \; \tilde{u}_i = \varphi \text{ on } \Gamma_{12}$$

$$\varphi \longrightarrow K_1 \frac{\partial \tilde{u}_1}{\partial n} - K_2 \frac{\partial \tilde{u}_2}{\partial n}$$

$$\mathcal{G} : -K_1 \cdot \frac{\partial u_1^0}{\partial n} + K_2 \cdot \frac{\partial u_2^0}{\partial n} \in H^{-1/2}(\Gamma_{12}) \; \Big\{ \; u_i^0 = 0 \text{ on } \Gamma_{12}$$

which leads to the following algebraic form

$$\mathcal{J} \cdot \varphi = \mathcal{G} \longrightarrow \boxed{\mathcal{J} = \mathcal{G} \cdot \varphi^{-1}}$$

of the Steklov-Poincare operator.

5 Operator Splitting Techniques

[1] $$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + a_x \frac{\partial u}{\partial x} = f$$

Discretization in space using Galerkin formulation & in time using BDF1

$$\int_\Omega \delta v \frac{\partial u}{\partial t} + \int_\Omega \frac{\partial \delta v}{\partial x} k \frac{\partial u}{\partial x} + \int_\Omega \delta v\, a_x \frac{\partial u}{\partial x} = \int_\Omega \delta v f$$

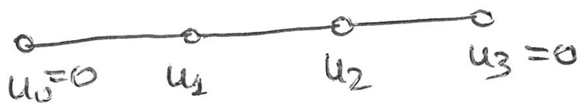
$$\frac{1}{\delta t}\int \delta v\, \Delta u + \int_\Omega \nabla \delta v\, k\, \nabla \delta u + \int_\Omega \delta v\, a_x \nabla u_x = \int_\Omega \delta v f$$

$$\frac{u^{n+1}-u^n}{\delta t}\int_\Omega N_i N_j + \left[\int_\Omega \nabla N_i\, k \nabla N_j + \int_\Omega N_i\, a_x \nabla N_j\right] u^{n+1} = \int_\Omega N_i f$$

algebraic form

$$M\frac{u^{n+1}-u^n}{\delta t} + \left[K + G\right]u^{n+1} = F^{n+1}$$

If we were to use 2node linear elements to discretize the 1D problem:



the elemental matrices for $k=1$, $a_x=1$, $f=1$,

$N_1(x) = \frac{x_2 - x}{x_2 - x}$ & $N_2(x) = \frac{x - x_1}{x_2 - x_1}$ are as follows:

$$K_e = \frac{1}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};\quad G_e = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix};\quad F_e = \frac{l^e}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

$$M_e = \frac{l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$ where $$\frac{U^1}{\delta t} = \begin{bmatrix} 0 & 10.99 & 4.33 & 0 \end{bmatrix}^T$$

⑤ Cont'd.

[2] we define $\mathcal{L} = \mathcal{L}_a + \mathcal{L}_K$

$$\mathcal{L}_a u = a_x \cdot \nabla u$$

$$\mathcal{L}_K u = -K \Delta u$$

which yields: $\partial_t u + \mathcal{L}_a u + \mathcal{L}_K u = f$

Then we introduce the splitting by defining intermediat variables : u_a & u_K

i) solving for $u_a(t_n) = u^n$

$$\partial_t u_a + \mathcal{L}_a u_a = 0$$

ii) then solving for u_K ~~with~~ with initial condition $u_a(t_{n+1})$

$$u_K(t_n) = u_a(t_{n+1})$$

$$\partial_t u_K + \mathcal{L}_K u_K = f$$

iii) to finally $u^{n+1} = u_K(t_{n+1})$

If we apply the above to the convection-diffusion equation we have:

i) $\frac{M}{\delta t} u_a^1 + C u_a^1 = 0 \xrightarrow{\text{solve}} \underline{u_a^1}$

ii) $\frac{M}{\delta t}(u_K^1 - u_a^1) + K u_K^1 = F^1 \xrightarrow{\text{solve}} u_K^1$

iii) $\boxed{u^1 = u_K^1}$

⑥ FRACTIONAL STEPS

[1] alpha appears on the third step which is where either pressure or velocity is corrected. I would answer that ~~the~~ an optimal value should be contained within the interval $(0,1)$.

[2] Yoside's scheme is characterized for ~~using~~ by replacing the exact LU factorization by an inexact LU factorization. This replacement introduces a splitting error for the velocity that is $O(\delta t^2)$ of second order.

⑦ ALE FORMULATIONS

[1] (a) Knowing that in the ALE framework

$$\frac{d}{dt}\gamma_{ALE}\left(\mathcal{X}(X,t), \mathcal{Y}(Y,t), \mathcal{Z}(z,t), t\right) = \frac{\partial \gamma_{ALE}(\dots)}{\partial t} + \nabla\gamma_{ALE} \cdot w$$

where:

$$\nabla\gamma_{ALE} \Rightarrow \nabla\gamma_{ALE} = \nabla\gamma \cdot \left(\frac{\partial x}{\partial X} + \frac{\partial y}{\partial Y} + \frac{\partial z}{\partial Z}\right)$$

$$\left(\frac{\partial x}{\partial X} + \frac{\partial y}{\partial Y} + \frac{\partial z}{\partial Z}\right) w = v - \text{mesh}.$$

yielding:

$$\boxed{\frac{d}{dt}\gamma_{ALE}(\dots) = \frac{\partial \gamma_{ALE}(\dots)}{\partial t} + (v - v_{mesh}) \cdot \nabla\gamma}$$

[1] (b) we do a first order time differentiation on the eq's of movement to obtain the velocity of the particles.

$$v = \frac{\partial(\circ)}{\partial t} = \begin{cases} v_x = X e^t \\ v_y = e^t \\ v_z = 0 \end{cases}$$

Same on the eq's of movement of the mesh to obtain the mesh velocity.

$$v_{mesh} = \begin{cases} v_{m,x} = \alpha \\ v_{m,y} = -\beta \\ v_{m,z} = 0. \end{cases}$$

8 FLUID-STRUCTURE INTERACTION

[1] The added mass effect is a problem of dynamic problems where inertial forces are introduced in the system. ~~For~~ fluid mechanics this effect appears when there is either a body or volume of fluid suffering some acceleration. ~~In~~ incompressible flows these effects can cause convergence issues.

The added mass effect can be circumvented using either Aitken relaxation scheme; adopting relatively small time-increments to minimize the contribution of the mass matrix, etc.

[2] Apply 2 iterations of the Aitken relaxation scheme to the iteration-by-subdomain scheme for the heat transfer problem

$$M\frac{(U^{n+1}-U^{n})}{\delta t}+KU^{n+1}=F^{n+1}\xrightarrow{\delta t=1\ \&\ U^{n}=0}$$

$$\rightarrow (M+K)\,U^{n+1}=F^{n+1}$$

...