Master's Degree Numerical Methods in Engineering

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Coupled Problems

Homeworks

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1 Transmission conditions

1.1 First exercise

The deflection $v(x)$ of an Euler-Bernoulli beam is governed by the differential equation

$$EI\,\frac{d^4v}{dx^4} = f$$

where EI is a mechanical property of the beam section and the beam material and f is the distributed load. Assuming for example that the beam is clamped at $x = 0$ and $x = L$, the Principle of Virtual Work (PTV) states that the solution $v(x)$ satisfies

$$EI\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2} = \int_0^L \delta v f$$

for all $\delta v(0) = \delta v(L) = 0$ and $\frac{d\delta v(0)}{dx} = \frac{d\delta v(L)}{dx} = 0$

(a) Postulate the space of functions where both v and δv must belong. Justify the answer.

(b) If $[0, L] = [0, P] \cup (P, L]$ obtain the transmission conditions at P implied by regularity requirements.

(c) Obtain the transmission conditions at P that follow by imposing in the PTV integral is additive.

1.1.1 First task

Postulate the space of functions where both v and δv must belong. Justify the answer.

Our first objective would be to find functions belonging to the space of functions whose integrals are bounded. These functions are the $L^2(\Omega)$ functions :

$$L^2(\Omega) = \{v(x) : \int_\Omega v^2\, d\Omega < \infty\}$$

Both u and δu functions must belong to the space of functions whose second derivatives can be integrated, these Sobolev spaces are known as $H^2(\Omega)$.

$$H^2(\Omega) = \{v \in L^2(\Omega) : |\nabla^2 v| \in L^2(\Omega)\}$$

Then if both δv and v belong to the $H^2(\Omega)$ subspace, then their product belongs to the same subspace, which is in L^2:

$$\int_\Omega \nabla^2\delta v \cdot \nabla^2 v\, d\Omega < \infty$$

1.1.2 Second task

If $[0, L] = [0, P] \cup (P, L]$ obtain the transmission conditions at P implied by regularity requirements.

The idea is to split the domain in two different portions, one from $[0, P]$ and the other one from $(P, L]$. It is important to state first, that physically there is no such discontinuity, therefore we expect no jump between these two portions.

$$\Omega_1 = [0, P]$$

$$\Omega_2 = (P, L]$$

The first transmission conditions for an $H^2(\Omega)$ continuous function is that the displacements and fluxes (rotations) in each side of the domain at a distance very small from the division say ε is equal in both sides:

$$[v] = 0$$

$$\left[\frac{dv}{dx}\right] = 0$$

$$[v] = \lim_{\varepsilon \to 0} \; v(P + \varepsilon \cdot n_1) - v(P + \varepsilon \cdot n_2)$$

$$\left[\frac{dv}{dx}\right] = \lim_{\varepsilon \to 0} \; \frac{dv(P + \varepsilon \cdot n_1) - dv(P + \varepsilon \cdot n_2)}{dx}$$

Both conditions imposed with strong continuity.

1.1.3 Third task

Obtain the transmission conditions at P that follow by imposing in the PTV integral is additive.

When considering the strong form of the problem, $EI \; d^4v/dx^4 = f$ for each of the domains:

$$\int_{\Omega_1} \delta v \cdot EI \; \frac{d^4 v}{dx^4} \; d\Omega_1 = \int_{\Omega_1} \delta v \; f \; d\Omega_1$$

Applying integration by parts:

$$EI \cdot \left[\int_{\Omega_1} \frac{d}{dx}\left(\delta v \; \frac{d^3 v(x)}{dx^3}\right) \; d\Omega - \int_{\Omega_1} \frac{d\delta v}{dx} \; \frac{d^3 v(x)}{dx^3} \; d\Omega\right] = \int_{\Omega_1} \delta v \; f \; d\Omega$$

Applying integration by parts again:

$$EI \cdot \left[\int_{\Omega_1} \frac{d}{dx}\left(\delta v \; \frac{d^3 v(x)}{dx^3}\right) \; d\Omega_1 - \int_{\Omega_1} \frac{d}{dx}\left(\frac{d\delta v}{dx} \; \frac{d^2 v(x)}{dx^2}\right) \; d\Omega_1 + \int_{\Omega_1} \frac{d^2 \delta v}{dx^2} \; \frac{d^2 v(x)}{dx^2} \; d\Omega_1\right] = \int_{\Omega_1} \delta v \; f \; d\Omega_1$$

Applying Gauss Divergence Theorem and considering that the flux in the boundary is zero as a problem datum, the only new boundary $\Gamma = \Gamma_{12}$:

$$EI \cdot \left[\int_{\Gamma_{12}} n_1 \left(\delta v \; \frac{d^3 v(x)}{dx^3}\right) \; d\Gamma_{12} - \int_{\Gamma_{12}} n_1 \left(\frac{d\delta v}{dx} \; \frac{d^2 v(x)}{dx^2}\right) \; d\Gamma_{12} + \int_{\Omega_1} \frac{d^2 \delta v}{dx^2} \; \frac{d^2 v(x)}{dx^2} \; d\Omega_1\right] = \int_{\Omega_1} \delta v \; f \; d\Omega_1$$

Applying the exact same procedure in Ω_2, adding both equations and stating $\Omega = \Omega_1 + \Omega_2$:

$$EI \cdot \left[\int_{\Omega} \frac{d^2 \delta v}{dx^2} \; \frac{d^2 v(x)}{dx^2} \; d\Omega - \int_{\Gamma_{12}} \left[n_1 \left(\delta v \; \frac{d^3 v(x)}{dx^3} + \frac{d\delta v}{dx} \; \frac{d^2 v(x)}{dx^2}\right) + n_2 \left(\delta v \; \frac{d^3 v(x)}{dx^3} + \frac{d\delta v}{dx} \; \frac{d^2 v(x)}{dx^2}\right)\right] \; d\Gamma_{12}\right] = \int_{\Omega} \delta v \; f \; d\Omega \quad (1)$$

From Eq. (1) it can be seen that the first term in the left and the only term in the right are the same as in problem statement equation, therefore they can be canceled out.

$$EI \cdot \int_{\Gamma_{12}} \left[n_1 \cdot \left(\delta v \; \frac{d^3 v(x)}{dx^3} + \frac{d\delta v}{dx} \; \frac{d^2 v(x)}{dx^2}\right) + n_2 \cdot \left(\delta v \; \frac{d^3 v(x)}{dx^3} + \frac{d\delta v}{dx} \; \frac{d^2 v(x)}{dx^2}\right)\right] \; d\Gamma_{12} = 0 \quad (2)$$

Given the fact that the '*virtual displacements*' are every arbitrary displacements that satisfy $\delta v(0) = \delta v(L) = 0$ and $d\delta v(0)/dx = d\delta v(L)/dx = 0$, which provides no restriction to any relevant Γ_{12} splitting of the domain.

$$EI \cdot \int_{\Gamma_{12}} \left[\delta v \cdot \left(n_1 \cdot \frac{d^3 v(x)}{dx^3} + n_2 \cdot \frac{d^3 v(x)}{dx^3}\right) + \frac{d\delta v}{dx} \cdot \left(n_1 \cdot \frac{d^2 v(x)}{dx^2} + n_2 \cdot \frac{d^2 v(x)}{dx^2}\right)\right] = 0 \quad (3)$$

Then, for the equality in Eq. (3) to hold, the discretization should be regular enough to accomplish $n = n_1 = -n_2$, which is granted for δv and v are $H^2(\Omega)$ functions, this condition holds. And the conditions obtained are:

$$n_1 \cdot \frac{d^3 v(x)}{dx^3} = n_2 \cdot \frac{d^3 v(x)}{dx^3} \quad (4)$$

$$n_1 \cdot \frac{d^2 v(x)}{dx^2} = n_2 \cdot \frac{d^2 v(x)}{dx^2} \quad (5)$$

where Eq. (4) is the continuity in shear-associated strains and Eq. (5) is the continuity of the curvature in the deformation which is related to the bending moment of the beam. Both conditions imposed with weak continuity.

1.2 Second exercise

The Maxwell problem consists in finding a vector field $\mathbf{u}: \ \Omega \rightarrow \mathbb{R}^3$ such that

$$\begin{aligned} \nu \ \nabla \times \nabla \times \mathbf{u} &= \mathbf{f} \qquad in \ \Omega \\ \nabla \cdot \mathbf{u} &= 0 \qquad in \ \Omega \\ \mathbf{n} \ \times \ \mathbf{u} &= 0 \qquad in \ \partial\Omega \end{aligned}$$

where $\nu > 0$, $\mathbf{f}$ is a divergence-free force field and $\mathbf{n}$ the unit external normal. Equation $\nabla \cdot \mathbf{u} = 0$ is in fact redundant.

1.2.1 First task

Write a variational statement of the problem. Postulate the space of functions where $\mathbf{u}$ *must belong. Justify the answer.*
To analyse the problem the first equation will be multiplied by a test function and integrate to get to its weak form as made in section 1.1.1 to find the minimum continuity of the functions.

$$\int_\Omega \delta\mathbf{u} \cdot (\nu \ \nabla \times \nabla \times \mathbf{u}) \ d\Omega = \int_\Omega \delta\mathbf{u} \cdot \mathbf{f} \, d\Omega$$

Using the expression of the divergence of a curl of two vectors w and v:

$$\nabla \cdot (v \times w) = w \cdot (\nabla \times v) - v \cdot (\nabla \times w)$$

Using v and w equal to:

$$v = \nabla \times \mathbf{u}$$

$$w = \delta\mathbf{u}$$

We get to the following expression:

$$\nabla \cdot (\nabla \times \mathbf{u} \times \delta\mathbf{u}) = (\nabla \times \mathbf{u}) \cdot (\nabla \times \delta\mathbf{u}) - \delta\mathbf{u} \cdot (\nabla \times \nabla \times \mathbf{u})$$

And re-writing it:

$$\delta\mathbf{u} \cdot (\nabla \times \nabla \times \mathbf{u}) = (\nabla \times \mathbf{u}) \cdot (\nabla \times \delta\mathbf{u}) - \nabla \cdot (\nabla \times \mathbf{u} \times \delta\mathbf{u})$$

The integral expression can be written as:

$$\int_\Omega \nu \ (\nabla \times \delta\mathbf{u}) \cdot (\nabla \times \mathbf{u}) \ d\Omega - \int_\Omega \nu \ \nabla \cdot (\delta\mathbf{u} \times \nabla \times \mathbf{u}) \ d\Omega = \int_\Omega \delta\mathbf{u} \cdot \mathbf{f} \, d\Omega$$

Using Gauss divergence theorem in the second part of the integral:

$$\int_\Omega \nu \ (\nabla \times \delta\mathbf{u}) \cdot (\nabla \times \mathbf{u}) \ d\Omega - \int_{\partial\Omega} \nu \ \mathbf{n} \cdot (\delta\mathbf{u} \times \nabla \times \mathbf{u}) \ d\partial\Omega = \int_\Omega \delta\mathbf{u} \cdot \mathbf{f} \, d\Omega \tag{6}$$

Using the vector product property: $X \cdot (Y \times Z) = Z \cdot (X \times Y) = Y \cdot (Z \times X)$ and applying it to the integral on the boundary considering $X = \mathbf{n}$, $Y = \delta\mathbf{u} \times \nabla$ and $Z = \mathbf{u}$:

$$\int_\Omega (\nabla \times \delta\mathbf{u}) \cdot (\nabla \times \mathbf{u}) d\Omega + \int_{\partial\Omega} (\delta\mathbf{u} \times \nabla) \cdot (\mathbf{u} \times \mathbf{n}) \ d\partial\Omega = \int_\Omega \delta\mathbf{u} \cdot \mathbf{f} \, d\Omega$$

where the product $\mathbf{n} \times \mathbf{u} = -\mathbf{u} \times \mathbf{n} = 0$ given that the test functions can be any function that comply the boundary conditions of the problem.
Finally we get to the expression in Eq. (7) from which the continuity conditions can be obtained.

$$\int_\Omega (\nabla \times \delta\mathbf{u}) \cdot (\nabla \times \mathbf{u}) \ d\Omega = \int_\Omega \delta\mathbf{u} \cdot \mathbf{f} \, d\Omega \tag{7}$$

Both $\mathbf{u}$ and $\delta\mathbf{u}$ need to belong to a space whose integrals are bounded. In this case, the operation being performed is not a dot product but a cross product.

$$L^2(\Omega) = \{u(x) : \int_\Omega ||\nabla \times \mathbf{u}||^2 \; d\Omega < \infty\}$$

Because the cross product involves only first order derivatives:

$$\nabla \times u = \left(\frac{\partial u(\mathbf{x})}{\partial x_1}; \frac{\partial u(\mathbf{x})}{\partial x_2}; ...; \frac{\partial u(\mathbf{x})}{\partial x_n} \right)$$

It is enough for the to belong to first order Hilbert spaces

Both $\mathbf{u}$ and $\delta\mathbf{u}$ functions must belong to the space of functions whose first derivatives can be integrated, these Sobolev spaces are known have to comply the continuity conditions of a cross-derivative of a normal $\mathbb{H}^1(\Omega)$ space, and we can name it as $\mathbb{H}^{Curl}(\Omega)$.

$$H^{Curl}(\Omega) = \{\mathbf{u} \in L^2(\Omega) : |\nabla \times \mathbf{u}| \in L^2(\Omega)\}$$

1.2.2 Second task

If Γ is a surface that intersects Ω, obtain the transmission conditions across Γ implied by regularity requirements. From problem statement we know the first transmission condition:

$$[\mathbf{n} \times \mathbf{u}]_\Gamma = 0$$

Imposed in a strong way. If we want to estimate how a this continuity affects the problem, a small region of interest may be considered $x_0 \pm \varepsilon$.

$$[\mathbf{n} \times \mathbf{u}] = \lim_{\varepsilon \to 0} \; [(\mathbf{n} \times u(x_0 + \varepsilon)) - (\mathbf{n} \times u(x_0 - \varepsilon))]$$

If this continuity does not hold, we can estimate the value of the jump and the values of each of the domains Ω_1 and Ω_2 as:

$$[\mathbf{n} \times \mathbf{u}] = \begin{cases} \nabla \times \mathbf{u}_1 & in\ \Omega_1 \\ (1/2 \cdot \varepsilon) \cdot [(\mathbf{n} \times \mathbf{u}(x_0 + \varepsilon)) - (\mathbf{n} \times \mathbf{u}(x_0 - \varepsilon))] & x_0 - \varepsilon \leq x \leq x_0 + \varepsilon \\ \nabla \times \mathbf{u}_2 & in\ \Omega_2 \end{cases}$$

Now integrating this expression:

$$\int_\Omega (\mathbf{n} \times \mathbf{u})^2 \; d\Omega = \int_{\Omega_1} (\mathbf{n} \times \mathbf{u}_1)^2 \; d\Omega_1 + \int_{x_0-\varepsilon}^{x_0+\varepsilon} (1/2 \cdot \varepsilon) \cdot [(\mathbf{n} \times \mathbf{u}(x_0 + \varepsilon)) - (\mathbf{n} \times \mathbf{u}(x_0 - \varepsilon))] \; dx + \int_{\Omega_2} (\mathbf{n} \times \mathbf{u}_1)^2 \; d\Omega_2$$

Considering additive integrals:

$$\int_\Omega (\mathbf{n} \times \mathbf{u})^2 \; d\Omega = \int_{\Omega_1+\Omega_2} (\mathbf{n} \times \mathbf{u})^2 \; d(\Omega_1 + \Omega_2) + \int_{x_0-\varepsilon}^{x_0+\varepsilon} (1/2 \cdot \varepsilon) \cdot [(\mathbf{n} \times \mathbf{u}(x_0 + \varepsilon)) - (\mathbf{n} \times \mathbf{u}(x_0 - \varepsilon))] \; dx$$

As LHS is equal to the first term of the LHS:

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} (1/2 \cdot \varepsilon) \cdot [(\mathbf{n} \times \mathbf{u}(x_0 + \varepsilon)) - (\mathbf{n} \times \mathbf{u}(x_0 - \varepsilon))] \; dx = 0$$

Which clearly states that if u is not continuous in the boundaries, the transmission condition is not complied (as $\varepsilon \to 0$ then $1/\varepsilon \to \infty$).

1.2.3 Third task

Obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive.

To obtain the transmission conditions the weak form of the problem equations are used in both of the domains.

$$\int_{\Omega_1} \nu\,(\nabla\times\delta\mathbf{u})\cdot(\nabla\times\mathbf{u})\ d\Omega_1 + \int_{\partial\Omega_1} \nu\,(\delta\mathbf{u}\times\nabla)\cdot(\mathbf{u}_1\times\mathbf{n}_1)\ d\partial\Omega_1 + \int_{\partial\Omega_{12}} \nu\,(\delta\mathbf{u}\times\nabla)\cdot(\mathbf{u}_1\times\mathbf{n}_1)\ d\partial\Omega_{12} = \int_{\Omega_1} \delta\mathbf{u}\cdot\mathbf{f} d\Omega_1 \tag{8}$$

$$\int_{\Omega_2} \nu\,(\nabla\times\delta\mathbf{u})\cdot(\nabla\times\mathbf{u})\ d\Omega_2 + \int_{\partial\Omega_2} \nu\,(\delta\mathbf{u}\times\nabla)\cdot(\mathbf{u}_2\times\mathbf{n}_2)\ d\partial\Omega_2 + \int_{\partial\Omega_{12}} \nu\,(\delta\mathbf{u}\times\nabla)\cdot(\mathbf{u}_2\times\mathbf{n}_2)\ d\partial\Omega_{12} = \int_{\Omega_1} \delta\mathbf{u}\cdot\mathbf{f} d\Omega_2 \tag{9}$$

Adding Eq. (8) and Eq. (9), considering that $\Omega_1 + \Omega_2 = \Omega$, the equations fulfill the initial weak form of the equations, therefore the terms $(\nabla\times\delta\mathbf{u})\cdot(\nabla\times\mathbf{u})\ d\Omega$ and $\delta\mathbf{u}\cdot\mathbf{f}\ d\Omega$ are cancelled out. Also, assuming $\partial\Omega_1+\partial\Omega_2 = \partial\Omega$ and imposing boundary boundary conditions ($\mathbf{u}\times\mathbf{n} = 0$). We finally get to the expression in Eq. (10)

$$\int_{\partial\Omega_{12}} \nu\,(\delta\mathbf{u}_1\times\nabla)\cdot(\mathbf{u}_1\times\mathbf{n}_1)\ d\partial\Omega_{12} + \int_{\partial\Omega_{12}} \nu\,(\delta\mathbf{u}_2\times\nabla)\cdot(\mathbf{u}_2\times\mathbf{n}_2)\ d\partial\Omega_{12} = 0$$

Re-writting the equation, using properties of cross-product:

$$\int_{\partial\Omega_{12}} \nu\ \delta\mathbf{u}_1\cdot(\nabla\times\mathbf{u}_1\times\mathbf{n}_1)\ d\partial\Omega_{12} + \int_{\partial\Omega_{12}} \nu\ \delta\mathbf{u}_2\cdot(\nabla\times\mathbf{u}_2\times\mathbf{n}_1)\ d\partial\Omega_{12} = 0 \tag{10}$$

$$[\nu\,(\nabla\times\mathbf{u}\times\mathbf{n})] = 0 \tag{11}$$

In Eq. (11) the second transmission condition is shown, and it is imposed weakly on the splitting boundary.

1.3 Third exercise

The Navier equations for an elastic material can be written in three different ways:

$$-2\mu\nabla\cdot(\varepsilon(\mathbf{u})) - \mu\nabla(\nabla\cdot\mathbf{u}) = \rho\mathbf{b}$$
$$-\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) = \rho\mathbf{b}$$
$$\mu\nabla\times(\nabla\times\mathbf{u}) - (\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) = \rho\mathbf{b}$$

where $\mathbf{u}$ is the displacement field, $\varepsilon(\mathbf{u})$ is the symmetric part of $\nabla\mathbf{u}$, λ and μ the Lamé coefficients, ρ the density of the material and $\mathbf{b}$ the body forces. Let us assume that $\mathbf{u} = 0$ on $\partial\Omega$.

1.3.1 First task

Write down the variational form of the previous equations in the appropriate functional spaces.

First equation:

The variational form of the equation considering both λ and μ constants:

$$\varepsilon(\mathbf{u}) = \nabla^S\mathbf{u} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla^T\mathbf{u}\right)$$

$$\int_\Omega -2\mu\ \mathbf{w}\ \nabla\cdot\nabla^S\mathbf{u}\ d\Omega - \int_\Omega \mathbf{w}\mu\nabla(\nabla\cdot\mathbf{u})\ d\Omega = \int_\Omega \mathbf{w}\rho\cdot\mathbf{b}d\Omega$$

Integrating first term by parts:

$$\int_\Omega \mathbf{w}\ \nabla\cdot\nabla^S\mathbf{u}\ d\Omega = \int_\Omega \nabla\cdot(\mathbf{w}\cdot\nabla^S\mathbf{u})\ d\Omega - \int_\Omega \nabla\mathbf{w} : \nabla^S\mathbf{u}\ d\Omega$$

Using Gauss Theorem:

$$\int_\Omega \mathbf{w}\ \nabla\cdot\nabla^S\mathbf{u}\ d\Omega = \int_\Gamma \mathbf{n}\cdot(\mathbf{w}\cdot\nabla^S\mathbf{u})\ d\Gamma - \int_\Omega \nabla\mathbf{w} : \nabla^S\mathbf{u}\ d\Omega$$

Integrating second term by parts:

$$\int_\Omega \mathbf{w}\ \nabla(\nabla\cdot\mathbf{u})\ d\Omega = \int_\Omega \nabla\cdot(\mathbf{w}\ (\nabla\cdot\mathbf{u}))\ d\Omega - \int_\Omega (\nabla\cdot\mathbf{w})\ (\nabla\cdot\mathbf{u})d\Omega$$

Using Gauss Theorem:

$$\int_\Omega \mathbf{w}\ \nabla(\nabla\cdot\mathbf{u})\ d\Omega = \int_\Gamma \mathbf{n}\cdot(\mathbf{w}\ (\nabla\cdot\mathbf{u}))\ d\Gamma - \int_\Omega (\nabla\cdot\mathbf{w})\ (\nabla\cdot\mathbf{u})d\Omega$$

Replacing in the general equation, and recalling that $\mathbf{u} = 0$ in $\Gamma = \partial\Omega$:

$$2\mu\int_\Omega \nabla\mathbf{w} : \nabla^S\mathbf{u}\ d\Omega + \mu\int_\Omega (\nabla\cdot\mathbf{w})\ (\nabla\cdot\mathbf{u})\ d\Omega = \int_\Omega \mathbf{w}\rho\cdot\mathbf{b}\ d\Omega \tag{12}$$

In Eq. (12) we determine by similarity of the operations involved (comparing with previous exercises gradients of functions) that both $\mathbf{w}$ and $\mathbf{u}$ must belong to the subspace $\mathbb{H}^1(\Omega)$.

<u>Second equation:</u>

$$-\int_\Omega \mathbf{w}\ \mu\ \Delta\mathbf{u}\ d\Omega - \int_\Omega \mathbf{w}\ (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})\ d\Omega = \int_\Omega \mathbf{w}\ \rho\ \mathbf{b}\ d\Omega$$

Applying integration by parts in the first term of the previous equation:

$$\int_\Omega \mathbf{w}\ \Delta\mathbf{u}\ d\Omega = \int_\Omega \mathbf{w}\ \nabla^2\mathbf{u}\ d\Omega = \int_\Omega \nabla\cdot(\nabla\mathbf{u}\cdot\mathbf{w})d\Omega - \int_\Omega \nabla\mathbf{w} : \nabla\mathbf{u}\ d\Omega$$

Using Gauss Theorem:

$$\int_\Omega \mathbf{w}\ \nabla\cdot\nabla\mathbf{u}\ d\Omega = \int_\Gamma \mathbf{n}\cdot(\nabla\mathbf{u}\cdot\mathbf{w})d\Gamma - \int_\Omega \nabla\mathbf{w} : \nabla\mathbf{u}\ d\Omega$$

Applying integration by parts in the second term of the equation:

$$\int_\Omega \mathbf{w}\ \nabla(\nabla\cdot\mathbf{u})\ d\Omega = \int_\Omega \nabla\cdot(\mathbf{w}\ (\nabla\cdot\mathbf{u}))\ d\Omega - \int_\Omega (\nabla\cdot\mathbf{w})(\nabla\cdot\mathbf{u})\ d\Omega$$

Using Gauss Theorem:

$$\int_\Omega \mathbf{w}\ \nabla(\nabla\cdot\mathbf{u})\ d\Omega = \int_\Gamma \mathbf{n}\cdot(\mathbf{w}\ (\nabla\cdot\mathbf{u}))\ d\Gamma - \int_\Omega (\nabla\cdot\mathbf{w})(\nabla\cdot\mathbf{u})\ d\Omega$$

Replacing in the general equation, and recalling that $\mathbf{u} = 0$ in $\Gamma = \partial\Omega$:

$$\int_\Omega \mu\ \nabla\mathbf{w} : \nabla\mathbf{u}\ d\Omega + \int_\Omega (\lambda+\mu)\ (\nabla\cdot\mathbf{w})(\nabla\cdot\mathbf{u})\ d\Omega = \int_\Omega \mathbf{w}\ \rho\mathbf{b}\ d\Omega \tag{13}$$

In Eq. (13) we determine that both $\mathbf{w}$ and $\mathbf{u}$ must belong to the subspace $\mathbb{H}^1(\Omega)$.

<u>Third equation:</u>

$$\int_\Omega \mu\ \nabla\times(\nabla\times\mathbf{u})\ d\Omega - \int_\Omega (\lambda+2\mu)\ \nabla\ (\nabla\cdot\mathbf{u})\ d\Omega = \int_\Omega \mathbf{w}\ \rho\mathbf{b}\ d\Omega$$

Before starting integrating by parts, in the first term a curl of the curl of a vector property will be used:

$$\nabla\times(\nabla\times\ X) = \nabla(\nabla\cdot X) - \Delta X = \nabla(\nabla\cdot X) - \Delta X = \nabla(\nabla\cdot X) - \nabla^2 X$$

The replacing in the initial equation:

$$\int_\Omega \mu \ \nabla(\nabla\cdot\mathbf{u})\ d\Omega - \int_\Omega \mu\nabla^2\mathbf{u}\ d\Omega - \int_\Omega (\lambda+2\mu)\ \nabla\ (\nabla\cdot\mathbf{u})\ d\Omega = \int_\Omega \mathbf{w}\ \rho\mathbf{b}\ d\Omega$$

Canceling the first term:

$$-\int_\Omega \mu\nabla^2\mathbf{u}\ d\Omega - \int_\Omega (\lambda+\mu)\ \nabla\ (\nabla\cdot\mathbf{u})\ d\Omega = \int_\Omega \mathbf{w}\ \rho\mathbf{b}\ d\Omega$$

Which is again the second equation, therefore the procedure is the same and so the results, shown in Eq. (13).

1.3.2 Second task

If Γ is a surface that intersects Ω, obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive.

First equation:

To obtain the *transmission conditions* of the equations the full expressions before taking into account the boundary conditions have to be used, as shown in Eq. (14) in each subdomain.

$$2\mu\int_\Omega \nabla\mathbf{w}:\nabla^S\mathbf{u}\ d\Omega + \int_\Gamma 2\mu\ \mathbf{n}\cdot(\mathbf{w}\cdot\nabla^S\mathbf{u})\ d\Gamma + \mu\int_\Omega (\nabla\cdot\mathbf{w})\ (\nabla\cdot\mathbf{u})\ d\Omega + \int_\Gamma \mu\mathbf{n}\cdot(\mathbf{w}\ (\nabla\cdot\mathbf{u}))\ d\Gamma = \int_\Omega \mathbf{w}\rho\cdot\mathbf{b}\ d\Omega \quad (14)$$

The boundary of the subdomains 1 and 2 are splitted into $\Gamma_i = \partial\Omega_i + \Gamma_{12}$, where $\partial\Omega_i$ belongs to the given Boundary conditions and Γ_{12} is the new boundary where the transmission conditions have to be obtained. The fact that $u = 0$ in $\partial\Omega_i$ is taken into account for simplicity.

Subdomain 1:

$$2\mu\int_{\Omega_1} \nabla\mathbf{w}_1:\nabla^S\mathbf{u}_1\ d\Omega_1 + \int_{\Gamma_{12}} 2\mu\ \mathbf{n}_1\cdot(\mathbf{w}_1\cdot\nabla^S\mathbf{u}_1)\ d\Gamma_{12} + \mu\int_{\Omega_1} (\nabla\cdot\mathbf{w}_1)\ (\nabla\cdot\mathbf{u}_1)\ d\Omega_1 + \int_{\Gamma_{12}} \mu\ \mathbf{n}_1\cdot(\mathbf{w}_1\ (\nabla\cdot\mathbf{u}_1))\ d\Gamma_{12} = \int_{\Omega_1} \mathbf{w}_1\rho\cdot\mathbf{b}\ d\Omega_1$$

Subdomain 2:

$$2\mu\int_{\Omega_2} \nabla\mathbf{w}_2:\nabla^S\mathbf{u}_2\ d\Omega_2 + \int_{\Gamma_{12}} 2\mu\ \mathbf{n}_2\cdot(\mathbf{w}_2\cdot\nabla^S\mathbf{u}_2)\ d\Gamma_{12} + \mu\int_{\Omega_2} (\nabla\cdot\mathbf{w}_2)\ (\nabla\cdot\mathbf{u}_2)\ d\Omega_2 + \int_{\Gamma_{12}} \mu\ \mathbf{n}_2\cdot(\mathbf{w}_2\ (\nabla\cdot\mathbf{u}_2))\ d\Gamma_{12} = \int_{\Omega_2} \mathbf{w}_2\rho\cdot\mathbf{b}\ d\Omega_2$$

Adding both expressions and considering the additive integrals:

$$2\mu\int_\Omega \nabla\mathbf{w}:\nabla^S\mathbf{u}\ d\Omega = 2\mu\int_{\Omega_1} \nabla\mathbf{w}_1:\nabla^S\mathbf{u}_1\ d\Omega_1 + 2\mu\int_{\Omega_2} \nabla\mathbf{w}_2:\nabla^S\mathbf{u}_2\ d\Omega_2$$

$$\mu\int_\Omega (\nabla\cdot\mathbf{w})\ (\nabla\cdot\mathbf{u})\ d\Omega = \mu\int_{\Omega_1} (\nabla\cdot\mathbf{w}_1)\ (\nabla\cdot\mathbf{u}_1)\ d\Omega_1 + \mu\int_{\Omega_2} (\nabla\cdot\mathbf{w}_2)\ (\nabla\cdot\mathbf{u}_2)\ d\Omega_2$$

$$\int_\Omega \mathbf{w}\rho\cdot\mathbf{b}\ d\Omega = \int_{\Omega_1} \mathbf{w}_1\rho\cdot\mathbf{b}\ d\Omega_1 + \int_{\Omega_2} \mathbf{w}_2\rho\cdot\mathbf{b}\ d\Omega_2$$

We get to:

$$\int_{\Gamma_{12}} 2\mu\ \mathbf{n}_1\cdot(\mathbf{w}_1\cdot\nabla^S\mathbf{u}_1)\ d\Gamma_{12} + \int_{\Gamma_{12}} \mu\ \mathbf{n}_1\cdot(\mathbf{w}_1\ (\nabla\cdot\mathbf{u}_1))\ d\Gamma_{12} + \int_{\Gamma_{12}} 2\mu\ \mathbf{n}_2\cdot(\mathbf{w}_2\cdot\nabla^S\mathbf{u}_2)\ d\Gamma_{12} + \int_{\Gamma_{12}} \mu\ \mathbf{n}_2\cdot(\mathbf{w}_2\ (\nabla\cdot\mathbf{u}_2))\ d\Gamma_{12} = 0$$

Considering that both w_i can be equal as long as they fulfill the Dirichlet boundary conditions:

$$\int_{\Gamma_{12}} \mathbf{w}(2\mu\ \mathbf{n}_1\cdot\nabla^S\mathbf{u}_1 + 2\mu\ \mathbf{n}_2\cdot\nabla^S\mathbf{u}_2)\ d\Gamma_{12}$$

$$\int_{\Gamma_{12}} \mathbf{w}(\mu\ \mathbf{n}_1\nabla\cdot\mathbf{u}_1 + \mu\ \mathbf{n}_2\nabla\cdot\mathbf{u}_2)\ d\Gamma_{12}$$

The transmission conditions weakly impposed are shown in Eq. (15) and (16).

$$[2\mu \; \mathbf{n} \cdot \nabla^S u] = 0 \tag{15}$$

$$[\mu \; \mathbf{n}(\nabla \cdot u)] = 0 \tag{16}$$

Second equation:

In this case, directly writing the non-zero terms in the addition of the splitting of subdomains 1 and 2, as shown in Eq. (17)

$$\int_{\Gamma_{12}} \mu \, \mathbf{n}_1 \, (\nabla \mathbf{u}_1 \cdot w_1) \, d\Gamma_{12} + \int_{\Gamma_{12}} (\lambda + \mu) \, \mathbf{n}_1 (\mathbf{w}_1 (\nabla \cdot \mathbf{u}_1) \, d\Gamma_{12} + \int_{\Gamma_{12}} \mu \, \mathbf{n}_2 \, (\nabla \mathbf{u}_2 \cdot w_2) \, d\Gamma_{12} + \int_{\Gamma_{12}} (\lambda + \mu) \, \mathbf{n}_2 (\mathbf{w}_2 (\nabla \cdot \mathbf{u}_2) \, d\Gamma_{12} \tag{17}$$

Considering the same weighting function in both subdomains:

$$\int_{\Gamma_{12}} \mathbf{w} \, (\mu \, \mathbf{n}_1 \cdot \nabla \mathbf{u}_1 + \mu \, \mathbf{n}_2 \cdot \nabla \mathbf{u}_2) \, d\Gamma_{12}$$

$$\int_{\Gamma_{12}} \mathbf{w} \, ((\lambda + \mu) \, \mathbf{n}_1 \, (\nabla \cdot \mathbf{u}_1) + (\lambda + \mu) \, \mathbf{n}_2 \, (\nabla \cdot \mathbf{u}_2)) \, d\Gamma_{12}$$

The transmission conditions weakly impposed are shown in Eq. (18) and (19).

$$[\mu \; \mathbf{n} \cdot \nabla u] = 0 \tag{18}$$

$$[(\mu + \lambda) \; \mathbf{n} \; (\nabla \cdot u)] = 0 \tag{19}$$

And the third equation has the same transmission conditions as shown to have the same equation than the second when using the curl property.

2 Domain decomposition methods

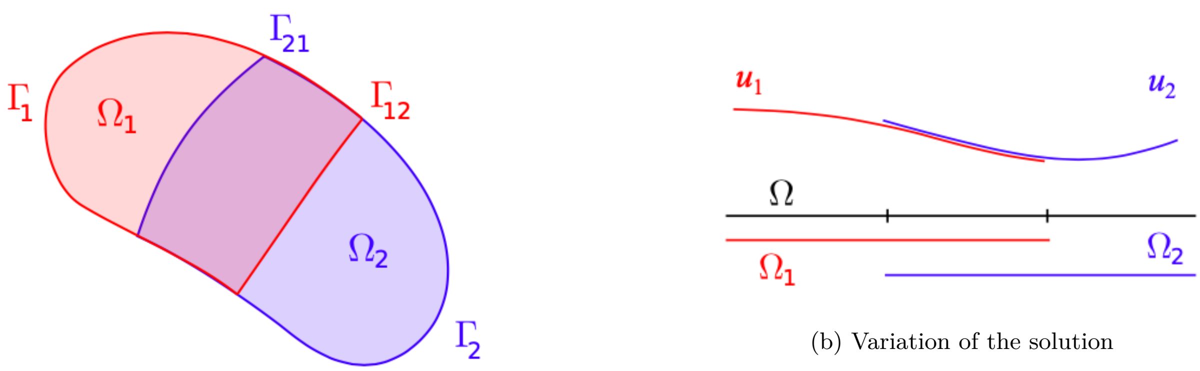
2.1 First exercise

Consider Problem 1 of Section 1. Let $[0, L] = [0, L_1] \cup [L_2, L]$*, with* $L_2 < L_1$*.*

2.1.1 First task

Write down an iteration-by-subdomain scheme based on a Schwarz additive domain decomposition method.

The Schwarz additive domain decomposition method is based on overlapping domain decomposition as seen in Figure 2a. Our case, is a 1D problem, and in Figure 2b closer geometry is shown, where $\Omega_1 = [0, L_1]$ and $\Omega_2 = [L_1, L_2]$.



(a) Split of the domain

(b) Variation of the solution

Figure 1: Overlapping domain decomposition.

In subdomain 1

$$
\begin{aligned}
EI\frac{d^4v_1^k(x)}{dx^4} &= f && \textit{in } \Omega \\
u_1^k &= \overline{u}_1 && \textit{on } \Gamma_1 \\
u_1^k &= u_2^{k-1} && \textit{on } \Gamma_{12}
\end{aligned}
$$

In subdomain 2

$$
\begin{aligned}
EI\frac{d^4v_2^k(x)}{dx^4} &= f && \textit{in } \Omega \\
u_2^k &= \overline{u}_2 && \textit{on } \Gamma_1 \\
u_2^k &= u_1^l && \textit{on } \Gamma_{12}
\end{aligned}
$$

where $l = k$ for Gauss-Seidel scheme and $l = k - 1$ for Jacobi.

2.1.2 Second task

Obtain the matrix version of the previous scheme once space has been discretized using finite elements.
Considering we have reached the expressions corresponding to the linear system of equations $A \cdot x = b$ where A represents the matrix version of the *strong form* of the problem of interest and x are the unknown nodal values we aim to find.
After splitting these domains, we can calculate the matrices corresponding to domain 1 (A_{11} in Ω_1) and domain 2 (A_{22} in Ω_2), also defining the values of the matrices in the boundaries ($A_{1\Gamma}$ in Ω_1 and $A_{2\Gamma}$ in Ω_2).

$$A_{ii} = R_i^T \cdot A \cdot R_i$$

with a restrictor operator R_i that suppress the whole space of functions $V^h\ (\Omega) \rightarrow V^h_{(i)}\ (\Omega)$.
Then the iterative scheme of the solution is:

$$
\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \cdot \begin{bmatrix} U_1^k \\ U_\Gamma^k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2} \cdot U_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)} \cdot U_\Gamma^{(k-1)} \end{bmatrix}
$$

$$A_{22} \cdot U_2^k = F_2 - A_{2\Gamma} \cdot U_\Gamma^l$$

where $l = k - 1$ for Jacobi scheme and $l = k$ for Gauss-Seidel.

2.2 Second exercise

Consider the problem 2 of Section 1. Let Γ be a surface that intersects Ω.

2.2.1 First task

Write down an iteration-by-subdomain scheme based on the Dirichlet-Neumann coupling.
The Maxwell problem consists in finding a vector field $\mathbf{u} : \Omega \rightarrow \mathbb{R}^3$ such that:

$$
\begin{cases}
\nu\ \nabla \times \nabla \times \mathbf{u} = \mathbf{f} & \textit{in } \Omega \\
\nabla \cdot u = 0 & \textit{in } \Omega \\
\mathbf{n} \times \mathbf{u} = \mathbf{0} & \textit{on } \Gamma
\end{cases}
$$

The iteration by domain consists in analysing the in both domains Dirichlet conditions and Neumman transmitted among them.

$$
[\Omega_1] = \begin{cases}
\nu \nabla \times \nabla \times \mathbf{u}_1^k = f_1 & \textit{in } \Omega_1 \\
\nabla \cdot \mathbf{u}_1^k = 0 & \textit{in } \Omega_1 \\
\mathbf{n}_1 \times \mathbf{u}_1^k = 0 & \textit{on } \Gamma_1 \\
\mathbf{n}_1 \times (\nabla \times \mathbf{u}_1^k) = \mathbf{n}_2 \times (\nabla \times \mathbf{u}_2^{k+1}) & \textit{on } \Gamma_{12}\ [\textit{Neumann BC}]
\end{cases}
$$

$$
[\Omega_2] = \begin{cases}
\nu \nabla \times \nabla \times \mathbf{u}_2^k = f_1 & \textit{in } \Omega_2 \\
\nabla \cdot \mathbf{u}_2^k = 0 & \textit{in } \Omega_2 \\
\mathbf{n}_2 \times \mathbf{u}_2^k = 0 & \textit{on } \Gamma_2 \\
\mathbf{n}_2 \times \mathbf{u}_2^k = \mathbf{n}_1 \times \mathbf{u}_1^k & \textit{on } \Gamma_{12}\ [\textit{Dirichlet BC}]
\end{cases}
$$

2.2.2 Second task

Obtain the expression of the Steklov-Poincaré operator of the problem.
To obtain the *Steklov-Poincaré* operator, a description of u is proposed as:

$$u_i = u_i^0 + \tilde{u}_i \ for \ i = 1,\ 2$$

Then, the problem is written as:

$$\begin{aligned} \nu\ \nabla\times\nabla\times\mathbf{u}_i^0 &= \mathbf{f}\ in\ \Omega_i & \nu\ \nabla\times\nabla\times\tilde{u}_i &= 0\ in\ \Omega_i \\ \nabla\cdot\mathbf{u}_i^0 &= \mathbf{f}\ in\ \Omega_i & \nabla\cdot\tilde{u}_i &= 0\ in\ \Omega_i \\ \mathbf{n}_i\times\mathbf{u}_i^0 &= \mathbf{0}\ in\ \Gamma_i & \mathbf{n}_i\times\tilde{u}_i &= \overline{u}_i\ in\ \Gamma_i \\ \mathbf{n}_i\times\mathbf{u}_i^0 &= \mathbf{0}\ in\ \Gamma_{12} & \mathbf{n}_i\times\tilde{u}_i &= \varphi\ in\ \Gamma_{12} \end{aligned}$$

The original problem is equivalent to this modification if φ complies: $u_i = u_i^0 + \tilde{u}_i = u|_{\Omega_i}$. To accomplish this, the second transmission condition obtained must be satisfied:

$$[\nu\ \mathbf{n}\,(\nabla\times\mathbf{u})] = 0$$

$$\nu\ \mathbf{n}\left(\nabla\times(\mathbf{u}_1^0+\tilde{u}_1)\right) = \nu\ \mathbf{n}\left(\nabla\times(\mathbf{u}_2^0+\tilde{u}_2)\right)$$

$$\nu\ \mathbf{n}\,(\nabla\times(\tilde{u}_1-\tilde{u}_2)) = \nu\ \mathbf{n}\left(\nabla\times(-\mathbf{u}_1^0+\mathbf{u}_2^0)\right)$$

Then we state: $\varphi = \nu\ \mathbf{n}\,(\nabla\times(\tilde{u}_1-\tilde{u}_2))$, and $G = \nu\ \mathbf{n}\left(\nabla\times(-\mathbf{u}_1^0+\mathbf{u}_2^0)\right)$. The continuity and injection of the functions is:

$$\mathrm{S} = H^{1/2}(\Gamma_{12}) \to H^{-1/2}(\Gamma_{12})$$

$$G \in H^{-1/2}(\Gamma_{12})$$

The *Steklov-Poincaré* operator for the problem reads:

$$\mathrm{S}\varphi = G$$

2.2.3 Third task

Obtain the matrix version of the previous scheme once space has been discretized using finite elements.
Considering Eq. (6) on both subdomains and imposing Dirichlet Boundary conditions of the whole domain, the matrices will be obtained from the following equations.

$$\int_{\Omega_1} \nu\,(\nabla\times\mathbf{w})\cdot(\nabla\times\mathbf{u})\ d\Omega_1 - \int_{\partial\Gamma_{12}} \nu\ \mathbf{w}\cdot(\nabla\times\mathbf{u}_1\times\mathbf{n}_1)\ d\Gamma_{12} = \int_{\Omega_1} \mathbf{w}\cdot\mathbf{f}\,d\Omega_1$$

$$\int_{\Omega_2} \nu\,(\nabla\times\mathbf{w})\cdot(\nabla\times\mathbf{u})\ d\Omega_2 - \int_{\partial\Gamma_{12}} \nu\ \mathbf{w}\cdot(\nabla\times\mathbf{u}_2\times\mathbf{n}_2)\ d\Gamma_{12} = \int_{\Omega_1} \mathbf{w}\cdot\mathbf{f}\,d\Omega_2$$

The *Dirichlet-Neumann* algorithm imposes the strongly Dirichlet conditions on Ω_2 and flux (Neumann) conditions weakly on Ω_1, and the algorithm is as follows.

$$\begin{bmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_\Gamma \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma \\ F_2 \end{bmatrix}$$

Integrating by subdomains, imposing Neumann conditions on Ω_1:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \cdot \begin{bmatrix} U_1^k \\ U_\Gamma^k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2}\cdot U_2^{k-1} - A_{\Gamma\Gamma}^{(2)}\cdot U_\Gamma^{k-1} \end{bmatrix}$$

And imposing Dirichlet conditions in Ω_2 with $l = k$ Gauss-Seidel and $l = k-1$ Jacobi scheme:

$$A_{22}\cdot U_2^k = F_2 - A_{2\Gamma}\cdot U_\Gamma^l$$

2.3 Third exercise

Consider the problem of finding $u: \ \Omega \to \mathbb{R}$ *such that*:

$$\begin{cases} -k\Delta u = f & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}$$

where $k > 0$. Let Γ be a surface crossing Ω.

2.3.1 First task

Write down an iteration-by-subdomain scheme based on the Dirichlet-Robin coupling.
The idea of the *Dirichlet-Robin* coupling requires the strong imposition of Dirichlet conditions in Ω_2 and the relaxed Robin conditions over the flux in Ω_1. Considering the problem being study it is known the transmission conditions as:

$$[u] = 0$$

$$\left[k\frac{\partial u}{\partial n}\right] = 0$$

Then the iterative scheme method, applying Robin conditions in Ω_1:

$$\begin{cases} -k_1\Delta u_1 = f & in\ \Omega_1 \\ u_1 = 0 & on\ \partial\Omega_1 \\ k_1 \cdot \frac{\partial u_1^k}{\partial n} + \gamma\ u_1^k = k_2 \cdot \frac{\partial u_2^{k-1}}{\partial n} + \gamma\ u_2^{k-1} & on\ \Gamma_{12} \end{cases}$$

And applying Dirichlet conditions in Ω_2, with l corresponding to Gauss-Seidel or Jacobi Schemes:

$$\begin{cases} -k_2\Delta u_2 = f & in\ \Omega_2 \\ u_2 = 0 & on\ \partial\Omega_2 \\ u_2^k = u_1^{(l)} & on\ \Gamma_{12} \end{cases}$$

2.3.2 Second task

Obtain the matrix version of the previous scheme once space has been discretized using finite elements.
The weak form of the problem will allow to calculate the matrices by means of the interpolation functions of the finite elements method.

$$\int_{\Omega_1} \nabla w \cdot (k\nabla u_1)\ d\Omega_1 - \int_{\Gamma_{12}} w\ (\mathbf{n}_1 \cdot \nabla u_1)\ d\Gamma_{12} = \int_{\Omega_1} w \cdot f\ d\Omega_1$$

$$\int_{\Omega_2} \nabla w \cdot (k\nabla u_2)\ d\Omega_2 - \int_{\Gamma_{12}} w\ (\mathbf{n}_2 \cdot \nabla u_2)\ d\Gamma_{12} = \int_{\Omega_2} w \cdot f\ d\Omega_2$$

Using these integrals and:

$$u \sim u^h(x) = \sum_{i=1}^{Nnodes} u_i \cdot N_i(x)$$

Same weighting functions and interpolation functions (Galerkin method):

$$w = N_j(x)$$

Then the matrices will be in the general shape of:

$$K_{jk}^{(i)} = \int_{\Omega^{(i)}} \nabla w \cdot (k\nabla u^{(i)})\ d\Omega^{(i)} = w^T \int_{\Omega^{(i)}} (\nabla N_j)^T \cdot (\nu\ \nabla N_k)\ d\Omega^{(i)} \cdot u^{(i)}$$

As stated previously, the domain Ω_1 applies the Robin conditions and the subdomain Ω_2 applies the Dirichlet boundary conditions, to clarify this the weak form of this coupling is written as.

$$\int_{\Omega_1} \nabla w \cdot (k\nabla u_1)\ d\Omega_1 - \int_{\Gamma_{12}} w\ (\mathbf{n}_2 \cdot \nabla u_2 + \gamma \cdot u_2^{k-1})\ d\Gamma_{12} = \int_{\Omega_1} w \cdot f\ d\Omega_1$$

$$\int_{\Omega_2} \nabla w \cdot (k \nabla u_2)\ d\Omega_2 = \int_{\Omega_2} w \cdot f\ d\Omega_2$$

Afterwards the matrix of the problem reads:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_\Gamma \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma \\ F_2 \end{bmatrix}$$

Integrating by subdomains, imposing Robin conditions on Ω_1:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \cdot \begin{bmatrix} U_1^k \\ U_\Gamma^k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - \ A_{\Gamma 2} \cdot U_2^{k-1} - A_{\Gamma\Gamma}^{(2)} \cdot U_\Gamma^{k-1} \end{bmatrix}$$

And imposing Dirichlet conditions in Ω_2 with $l = k$ Gauss-Seidel and $l = k - 1$ Jacobi scheme:

$$A_{22} \cdot U_2^k = F_2 - A_{2\Gamma} \cdot U_\Gamma^l$$

2.3.3 Third task

Obtain the Schur complement as discrete version of the Steklov-Poincaré operator.

$$\mathrm{S} = H^{1/2}(\Gamma_{12}) \to H^{-1/2}(\Gamma_{12})$$

$$G \in H^{-1/2}(\Gamma_{12})$$

The *Steklov-Poincaré* operator for the problem reads:

$$\mathrm{S}\varphi = G$$

where S and G are obtained by finding the values of U_1 and U_2 from first and third equations of the general matrix in the previous task, and replacing in the second equation:

$$\mathrm{S} = \left(A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}\right)$$

$$G = F_\Gamma - A_{\Gamma 1} A_{11}^{-1}\ F_1 - A_{\Gamma 2} A_{22}^{-1}\ F_2$$

2.3.4 Fourth task

Identify the preconditioner for the Schur complement equation arising from the iterative scheme of section (2.3.1).
To obtain the preconditioner for the Dirichlet-Robin the Schur matrix is written as:

$$S = S_1 + S_2$$

$$S_i = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma\ i}^{(i)} (A_{i\ i}^{-1}) A_{i\ \Gamma}$$

Considering the Gauss-Seidel scheme, the first and third equations of the general matrix are:

$$U_1^k = A_{11}^{-1} \cdot (F_1 - A_{1\Gamma}\ U_\Gamma^k)$$

$$U_2^{k-1} = A_{22}^{-1} \cdot (F_2 - A_{2\Gamma}\ U_\Gamma^{k-1})$$

Replacing these values in the second equation:

$$A_{\Gamma 1} U_1^k + A_{\Gamma\Gamma}^{(1)} U_\Gamma^k = F_\Gamma - A_{\Gamma 2} U_2^{k-1} - A_{\Gamma\Gamma}^{(2)} U_2^{k-1}$$

$$A_{\Gamma 1} A_{11}^{-1} \cdot (F_1 - A_{1\Gamma}\ U_\Gamma^k) + A_{\Gamma\Gamma}^{(1)} U_\Gamma^k = F_\Gamma - A_{\Gamma 2} A_{22}^{-1} \cdot (F_2 - A_{2\Gamma}\ U_\Gamma^{k-1}) - A_{\Gamma\Gamma}^{(2)} U_\Gamma^{k-1}$$

$$(A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma})\ U_\Gamma^k = (F_\Gamma - A_{\Gamma 1} A_{11}^{-1} F_1 - A_{\Gamma 2} A_{22}^{-1} F_2) - (A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}\ U_\Gamma^k) U_\Gamma^{k-1}$$

$$S_1\ U_\Gamma^k = G - S_2 U_\Gamma^{k-1}$$

$$S_1\ U_\Gamma^k = G - (S - S_1) U_\Gamma^{k-1}$$

We get to the iterative scheme:

$$U_\Gamma^k = U_\Gamma^{k-1} + S_1^{-1} \cdot (G - S \cdot U_\Gamma^{k-1})$$

That is a Richardson iteration for the Schur complement with preconditioner $P = S_1$.

3 Coupling heterogeneous problems

3.1 First exercise

Consider the beam described in Problem 1 of Section 1. Apart from being clamped at $x = 0$ and $x = L$, the beam is supported on an elastic wall that occupies the square $[0, L]$ x $[-L, 0]$, where $y = 0$ corresponds to the beam axis. The wall is clamped everywhere except on the upper wall, where the beam is. The wall displacements in the $x-$ and $y-$directions are u and v, respectively, and the elastic properties E (Young modulus) and ν (Poisson's coefficient). No loads are applied on the wall, except for those coming from the beam.

3.1.1 First task

Write down the equations in the wall assuming a plane stress behavior.
Considering elastic wall behaviour it can be written:

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}$$

Then the plane-stresses description is:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

The momentum equation for the problem reads:

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{b} = 0$$

where $\underline{\text{b}}$ represents the body forces of the problem.
For the weak form of the problem, the boundary conditions are everywhere Dirichlet but in the top where Neumann boundary conditions of normal traction are applied.

3.1.2 Second task

Write down the equations for the beam modified because of the presence of the wall.
The general Euler-Bernoulli equations now needs to take into account the reaction of the elastic deformable wall which will be related to the vertical deformation v:

$$EI\ \frac{d^4 v}{dx^4} = f - R(v)$$

$$EI\ \frac{d^4 v}{dx^4} = f - \sigma_Y^{wall} \cdot v$$

the boundary conditions for the beam remain unchanged $v(0) = v(L) = dv(0)/dx = dv(L)/dx = 0$.

3.1.3 Third task

Obtain the adequate transmission conditions for v and the normal component of the traction on the wall at $y = 0$.
The transmission conditions to couple a Bernoulli-Euler with an elastic wall will require the strong imposition of displacements and rotations and the weak imposition of bending moment and shear forces.
First transmission condition for v reads:

$$v_B(x) = v_W(x, y = 0)$$

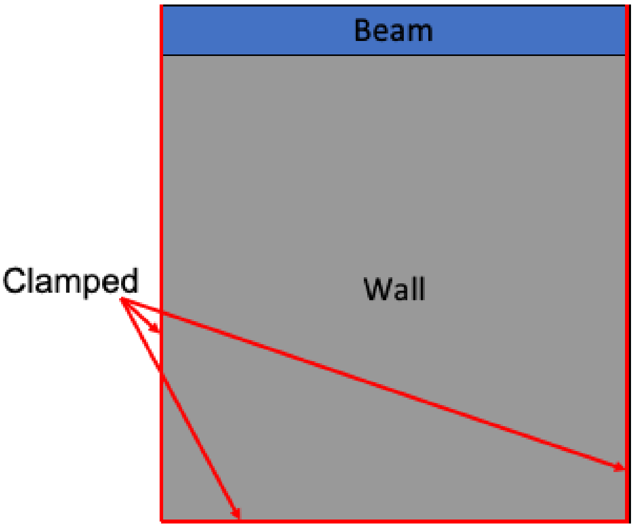
Afterwards, the tractions between the bottom face of the beam and the upper face of the wall, in contact need to ensure weakly. For this end the integration of the strong form describing the wall behaviour is shown:

$$\int_\Omega \underline{w}\ \nabla \cdot \underline{\underline{\sigma}}\ d\Omega + \int_\Omega \underline{w} \cdot \underline{b}\ d\Omega = 0$$

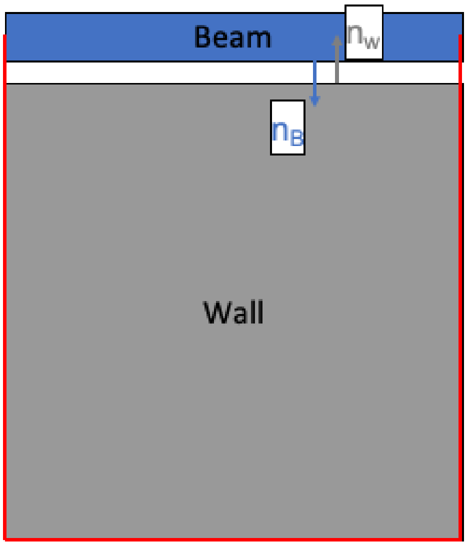
$$\int_{\Omega} \nabla \underline{w} \cdot \underline{\underline{\sigma}} \, d\Omega - \int_{\Gamma} \underline{w} \cdot (\underline{\underline{\sigma}} \cdot \underline{n}) \, d\Gamma = \int_{\Omega} \underline{w} \cdot \underline{b} \, d\Omega$$

Then the second transmission condition for the system is:

$$\tau_B \cdot \underline{n}_B = \underline{\underline{\sigma}}_w \cdot \underline{n}_w$$



(a) Scheme



(b) Normals

Figure 2: Beam+Wall structure.

3.1.4 Fourth task

Suggest transmission conditions for u and the tangent component of the traction on the wall at $y = 0$. Discuss the implications if this component is not assumed to be zero.

Since no friction is considered between the elements the displacements in the *x-direction* are different, and moreover the tractions are also different.

$$[u(x)] \neq 0$$

$$[\tau \cdot \underline{n}_x] \neq 0$$

If these components are not zero, a friction term will appear and therefore the u in the wall and in the beam will be equal, and will generate a flux of stresses between the structures.

3.2 Second exercise

Let S_D and S_S be the Dirichlet-to-Neumann operators for the Darcy and the Stokes problems, respectively (see the class notes, chapter 3). The Steklov-Poincaré equation can be written as

$$S_S(\lambda) = S_D(\lambda)$$

where λ is the velocity on Γ, the interface between Darcy and the Stokes region.

3.2.1 First task

Obtain the discrete version of the previous equation when space is discretized using finite elements. Relate the resulting matrices to those arising from the discretization of the Darcy and the Stokes problems separately.

In Figure 3 an sketch of the problem is shown. Also, each of the differential equations in strong form are are shown.

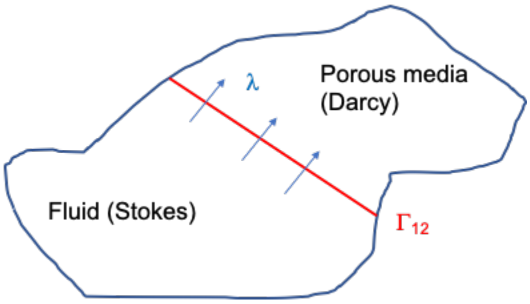


Figure 3: Stokes - Darcy coupled problem.

Stokes problem:

$$\begin{cases} -\nu\ \Delta\mathbf{u}_S + \nabla p_S = \mathbf{f} & in\ \Omega_S \\ \nabla\cdot\mathbf{u}_S = 0 & in\ \Omega_S \\ \mathbf{u}_S = \overline{\mathbf{u}}_S & on\ \Gamma_S \end{cases}$$

Darcy problem:

$$\begin{cases} \mathbf{u}_D + \underline{\underline{K}}\cdot\nabla\varphi = 0 & in\ \Omega_D \\ \nabla\cdot\mathbf{u}_D = 0 & in\ \Omega_D \\ \mathbf{n}\cdot\mathbf{u}_s = \overline{u}_{n,D} & on\ \Gamma_D \end{cases}$$

Deriving first the weak form of the Stokes problem:

$$\int_{\Omega_S} \mathbf{w}\cdot(-\nu\Delta\mathbf{u}_S + \nabla p_S)\ d\Omega_S = \int_{\Omega_S} \mathbf{w}\cdot\mathbf{f}\ d\Omega_S$$

$$\int_{\Omega_S} \mathbf{w}\cdot\nabla\mathbf{u}_S\ d\Omega_S = 0$$

Integrating by parts and using Divergence theorem:

$$\int_{\Omega_S} \nu\ \nabla\mathbf{w} : \nabla\mathbf{u}_S\ d\Omega_S - \int_{\Omega_S} p_S\ \nabla\cdot\mathbf{w}\ d\Omega_S - \int_{\Gamma_S} \mathbf{w}\cdot[\mathbf{n}\cdot(-p_S\cdot\underline{\underline{I}} + \nu\nabla\mathbf{u}_S)]\ d\Gamma = \int_{\Omega_S} \mathbf{w}\cdot\mathbf{f}\ d\Omega_S$$

Deriving the weak form of the Darcy problem:

$$\int_{\Omega_D} \mathbf{p}\cdot(\underline{\underline{K}}^{-1}\cdot u_D + \nabla\varphi)\ d\Omega_D = 0$$

Integrating by parts and using Divergence theorem:

$$\int_{\Omega_D} \mathbf{p}\cdot\underline{\underline{K}}^{-1}\cdot u_D\ d\Omega_D + \int_{\Omega_D} \varphi\ \nabla\cdot\mathbf{p}\ d\Omega_D - \int_{\Gamma} \varphi\ \mathbf{p}\cdot\mathbf{n}\ d\Gamma = 0$$

The continuity of fluxes is:

$$\mathbf{n}\cdot\mathbf{u}_s = \mathbf{n}\cdot\mathbf{u}_D$$

Considering a 2-dimension problem:

$$\mathbf{w}_S = w_S^n\mathbf{n} + w_S^\tau\mathbf{t}$$

$$\mathbf{p}_D = p_D^n\mathbf{n} + p_D^\tau\mathbf{t}$$

Then using these expressions on the flux boundary conditions in Stokes problem:

$$\mathbf{w}_S\cdot[\mathbf{n}\cdot(p_s\underline{\underline{I}} - \nu\nabla\mathbf{u}_S)] = p_s w_S^n - \nu\ \mathbf{n}\cdot\nabla\mathbf{u}_S\cdot\mathbf{n}\ w_S^n - \nu\ \mathbf{n}\cdot\nabla\mathbf{u}_S\cdot\mathbf{t}\ w_S^t$$

Doing the same for Darcy's problem:

$$\varphi\ \mathbf{n}\cdot\mathbf{p}_D = \varphi\ p_D^n$$

Now, the continuity of normal tractions reads, considering for any two weighting functions accomplishing Dirichlet Boundary conditions:

$$\varphi = p_s - \nu\ \mathbf{n} \cdot \nabla \mathbf{u}_S \cdot \mathbf{n}$$

The tangential component of the velocity does not need to be continuous, commonly the Balled-Joseph transmission condition is imposed as:

$$\nu\ \mathbf{t} \cdot (\mathbf{n} \cdot \nabla \mathbf{u}_s) = -\tfrac{\alpha_{BJ}}{\sqrt{K}}(\mathbf{u}_S - \mathbf{u}_D) \cdot \mathbf{t} \quad on\ \Gamma$$

It is considered that $|\mathbf{u}_D| \ll |\mathbf{u}_S|$ and the multiplication $(\mathbf{u}_s - \mathbf{u}_D) \cdot \mathbf{t} \approx \mathbf{u}_s \cdot \mathbf{t}$. Finally, the transmission conditions on Γ are:

$$\mathbf{n} \cdot \mathbf{u}_s = \mathbf{n} \cdot \mathbf{u}_D$$

$$p_s - \nu\ \mathbf{n} \cdot \nabla \mathbf{u}_S \cdot \mathbf{n} = \varphi$$

$$\mathbf{u}_s \cdot \mathbf{t} + \frac{\sqrt{K}}{\alpha_{BJ}}\ \nu\ \mathbf{t} \cdot (\mathbf{n} \cdot \nabla \mathbf{u}_s) = 0$$

These expressions into the weak forms would be:

$$\int_{\Omega_S} \nu\ \nabla \mathbf{w}_S : \nabla \mathbf{u}_S\ d\Omega_S - \int_{\Omega_S} p_S\ \nabla \cdot \mathbf{w}_S\ d\Omega_S - \int_{\Gamma_S} \mathbf{w}_S^n \cdot (p_s - \nu\ \mathbf{n} \cdot \nabla \mathbf{u}_S \cdot \mathbf{n})\ d\Gamma - \int_{\Gamma_S} \mathbf{w}_S^t \cdot (\mathbf{u}_s \cdot \mathbf{t} + \frac{\sqrt{K}}{\alpha_{BJ}}\ \nu\ \mathbf{t} \cdot (\mathbf{n} \cdot \nabla \mathbf{u}_s))\ d\Gamma$$

$$= \int_{\Omega_S} \mathbf{w}_S \cdot \mathbf{f}\ d\Omega_S$$

And the incompressibility:

$$-\int_{\Omega_S} q\ \nabla \cdot u_S\ d\Omega_S = 0$$

The finite element representations of the variables of interest will be of the shape shown as follows, considering Galerkin:

$$\mathbf{u}_S \approx u_S^h = \sum_{i=0}^{n_{dof}} N_i \cdot \tilde{U}_{S_i}^{sd=i} \cdot \mathbf{e}_i$$

$$p_S \approx p_S^h = \sum_{i=0}^{n_{dof}} \tilde{N}_i \cdot \tilde{P}_{S_i}^{sd}$$

$$\mathbf{w}_S = \sum_{i=0}^{n_n} N_i\ \tilde{W}_{S_i} \cdot \mathbf{e}_i$$

$$q_S = \tilde{N}_i\ Q_S$$

Darcy approximations:

$$\mathbf{u}_D \approx u_D^h = \sum_{i=0}^{n_{dof}} N_i \cdot \tilde{U}_{D_i}^{sd=i} \cdot \mathbf{e}_i$$

$$\varphi_D \approx \varphi_D^h = \sum_{i=0}^{n_{dof}} \tilde{N}_i \cdot \tilde{\mathbf{\Phi}}_{D_i} \cdot \mathbf{e}_i$$

$$\mathbf{w}_D = \sum_{i=0}^{n_n} N_i$$

$$q_D = \tilde{N}_i\ Q_D$$

Using these expressions, the problems can be expressed in a discrete manner using the matrix descriptions first for the Stokes problem:

$$\begin{bmatrix} A_{SS} & A_{S\Gamma} & G_{SS} \\ A_{\Gamma S} & A_{\Gamma\Gamma} & G_{\Gamma S} \\ G_{SS}^T & G_{S\Gamma}^T & 0 \end{bmatrix} \begin{bmatrix} U_S \\ \lambda \\ P_S \end{bmatrix} = \begin{bmatrix} f_{SS} \\ f_{S\Gamma} \\ h_S \end{bmatrix}$$

The Darcy problem:

$$\begin{bmatrix} A_{DD} & A_{D\Gamma} & G_{DD} \\ A_{\Gamma D} & A_{\Gamma\Gamma} & G_{\Gamma D} \\ G_{DD}^T & G_{D\Gamma}^T & 0 \end{bmatrix} \begin{bmatrix} U_D \\ \lambda \\ \Phi_D \end{bmatrix} = \begin{bmatrix} f_{DD} \\ f_{D\Gamma} \\ h_D \end{bmatrix}$$

It is possible to write in a single matrix system:

$$\begin{bmatrix} A_{SS} & G_{SS} & A_{S\Gamma} & 0 & 0 \\ G_{SS}^T & 0 & G_{\Gamma S} & 0 & 0 \\ A_{\Gamma S} & G_{S\Gamma} & A_{\Gamma\Gamma}^{(S)} + A_{\Gamma\Gamma}^{(D)} & A_{\Gamma D} & G_{D\Gamma} \\ 0 & 0 & A_{D\Gamma} & A_{DD} & G_{DD} \\ 0 & 0 & G_{D\Gamma}^T & G_{DD}^T & 0 \end{bmatrix} \begin{bmatrix} U_S \\ P_S \\ \lambda \\ U_D \\ \Phi_D \end{bmatrix} = \begin{bmatrix} f_{SS} \\ h_S \\ f_{S\Gamma} + f_{D\Gamma} \\ f_{DD} \\ h_D \end{bmatrix}$$

3.2.2 Second task

Write down the matrix form of a Dirichlet-Neumann iteration-by-subdomain using the matrices of the Darcy and the Stokes problems.
The iterative scheme reads:

$$\begin{bmatrix} A_{SS} & A_{S\Gamma} & G_{SS} \\ A_{\Gamma S} & A_{\Gamma\Gamma} & G_{\Gamma S} \\ G_{SS}^T & G_{S\Gamma} & 0 \end{bmatrix} \begin{bmatrix} U_S^k \\ \lambda^k \\ P_S^k \end{bmatrix} = \begin{bmatrix} f_{SS} \\ f_{S\Gamma} - A_{DD} \cdot \lambda^{k-1} - A_{\Gamma D} U_D^{k-1} \\ h_S \end{bmatrix}$$

And the Darcy part of the algorithm:

$$\begin{bmatrix} A_{DD} & G_{DD} \\ G_{DD}^T & 0 \end{bmatrix} \begin{bmatrix} U_D^k \\ \Phi_D^k \end{bmatrix} = \begin{bmatrix} f_{DD} - A_{D\Gamma}\lambda^{(l)} \\ h_D \end{bmatrix}$$

where again, $l = k$ for Gauss-Seidel and $l = k - 1$ for Jacobi.

3.2.3 Third task

Identify the Richardson iteration for the algebraic problem in section 3.2.1 resulting from section 3.2.2.
The Richardson iteration resulting from previous sections, have the shape: $u^{k+1} = u^k + (G - S\ U_\Gamma^k)$. In this way, using the previous expressions:

$$\begin{bmatrix} U_S^k \\ \lambda^k \\ P_S^k \end{bmatrix} = \begin{bmatrix} U_S^{k-1} \\ \lambda^{k-1} \\ P_S^{k-1} \end{bmatrix} + \left(\begin{bmatrix} f_{SS} \\ F_\Gamma - A_{\Gamma\Gamma}^{(D)}\ \lambda^{k-1} - A_{\Gamma D} U_D^{k-1} \\ h_S \end{bmatrix} - \begin{bmatrix} A_{SS} & A_{S\Gamma} & G_{SS} \\ A_{\Gamma S} & A_{\Gamma\Gamma} & G_{\Gamma S} \\ G_{SS}^T & G_{S\Gamma} & 0 \end{bmatrix} \begin{bmatrix} U_S^{k-1} \\ \lambda^{k-1} \\ P_S^{k-1} \end{bmatrix} \right)$$

Then, the Darcy part of the problem can be written as:

$$\begin{bmatrix} U_D^k \\ \Phi_D^k \end{bmatrix} = \begin{bmatrix} U_D^{k-1} \\ \Phi_D^{k-1} \end{bmatrix} + \left(\begin{bmatrix} f_{DD} - A_{D\Gamma}\lambda^{(l)} \\ h_D \end{bmatrix} - \begin{bmatrix} A_{DD} & G_{DD} \\ G_{DD}^T & 0 \end{bmatrix} \begin{bmatrix} U_D^{k-1} \\ \Phi_D^{k-1} \end{bmatrix} \right)$$

4 Monolithic and partitioned schemes in time

4.1 First exercise

Consider the one-dimensional, transient, heat transfer equation:

$$\begin{aligned} &\tfrac{\partial u}{\partial t} - k\ \tfrac{\partial^2 u}{\partial x^2} = f \quad in\ [0,1] \\ &u(x=0,\ t) = 0 \\ &u(x=1,\ t) = 0 \\ &u(x,\ t=0) = 0 \end{aligned}$$

4.1.1 First task

Discretize it using the finite element method (linear elements, element size h) for the discretization in space, and a $BDF1$ scheme for the discretization in time. Write down the weak form of the problem and the resulting matrix form of the problem, including the corresponding boundary integrals if necessary. Consider $\kappa = 1$, $f = 1$, $\delta t = 1$.

The time integration with discretization on time using θ method is:

$$\frac{\Delta u}{\Delta t} - \theta \Delta u_t = \partial_t u^n$$

where $\Delta u = u^{n+1} - u^n$ and $\Delta u_t = \partial_t u^{n+1} - \partial_t u^n$.
Replacing these scheme into the governing equation:

$$\frac{\Delta u}{\Delta t} + \theta \left(-k \frac{\partial^2 u^{n+1}}{\partial x^2} - \theta \ f^{n+1} \right) + (1-\theta) \left(-k \frac{\partial^2 u^n}{\partial x^2} - \theta \ f^n \right) = 0 \tag{20}$$

The previous equation allow to calculate the time integration, now it is necessary to integrating spatially the governing equations.

$$\int_\Omega v \cdot \left(\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} \right) \ d\Omega = \int_\Omega v \cdot f \ d\Omega$$

Integrating by parts:

$$\int_\Omega v \cdot \left(-k \frac{\partial^2 u}{\partial x^2} \right) \ d\Omega = - \int_\Omega \frac{\partial}{\partial x} \cdot \left(v \cdot k \ \frac{\partial u}{\partial x} \right) \ d\Omega + \int_\Omega \frac{\partial v}{\partial x} k \frac{\partial u}{\partial x} \ d\Omega$$

Replacing in the equation and using the divergence theorem:

$$\int_\Omega v \cdot \frac{\partial u}{\partial t} \ d\Omega - \int_\Gamma v \cdot \left(k \ \frac{\partial u}{\partial x} \cdot n \right) \ d\Gamma + \int_\Omega \frac{\partial v}{\partial x} k \frac{\partial u}{\partial x} \ d\Omega = \int_\Omega v \cdot f \ d\Omega$$

This expression can be written as, describing the spatial discretization of the problem.

$$(v, \ \partial_t u) + k(\nabla v, \ \nabla u) - < v, \ \mathbf{n} \cdot \nabla u >_\Gamma - < v, \ f >= 0$$

Replacing this equation into the time integration scheme in Eq. (20) considering Dirichlet boundary conditions, the expression reads:

$$\left(v, \frac{\Delta u}{\Delta t} \right) + \theta \cdot (k(\nabla v, \ \nabla u^{n+1}) - < v, \ f^{n+1} >) + (1-\theta) \cdot (k(\nabla v, \ \nabla u^n) - < v, \ f^n >) = 0$$

Considering the Finite Element + Galerkin description, the above expressions can be written in terms of different matrices $K, \ M$:

$$K = \int_\Omega k \ \nabla v \cdot \nabla u \ d\Omega \approx \sum_{i=1}^{n} \sum_{j=1}^{n} k \ \nabla N_i^T \cdot \nabla N_j \ w_{gp}$$

$$M = \int_\Omega v \cdot u \ d\Omega \approx \sum_{i=1}^{n} \sum_{j=1}^{n} k \ N_i^T \cdot N_j \ w_{gp}$$

which allows to write the complete $BDF1$ and considering f a constant source term, $\theta = 1$:

$$(M + \Delta t \ K) \cdot \Delta u =< v, f^{n+1} >$$

$$(M + \Delta t \ K) \ u^{k+1} =< v, f^{n+1} > + M u^k + \Delta t \ K \ u^k$$

This is the iterative scheme, and the matrices and vectors will be replaced for $A = (M + \Delta t \ K)$, $B =< v, f^{n+1} > + M u^k + \Delta t \ K \ u^k$.

$$u^{k+1} = A^{-1} \cdot B$$

4.1.2 Second task

Consider a domain decomposition approach for the previous problem. The left subdomain is composed of 2 elements ($h = 0.2$), while the right subdomain is composed of 3 elements ($h = 0.2$). Show that, if a monolithic approach is adopted, no boundary integrals are required at the interface. From now on, we denote the values at the nodes of the mesh as $u_0,\ u_1,\ u_2,\ u_3,\ u_4,\ u_5$. The interface is at u_2.

Considering the split of domain given in the exercise, the weak form of the problem reads:

$$\left(v, \tfrac{\partial u}{\partial t}\right)_{\Omega_1} + k(\nabla v,\ \nabla u)_{\Omega_1} - k\ < v,\ \mathbf{n}_1 \cdot \nabla u >_{\Gamma_{12}} - < v,\ f >_{\Omega_1} = 0 \quad in\ \Omega_1$$

$$\left(v, \tfrac{\partial u}{\partial t}\right)_{\Omega_1} + k(\nabla v,\ \nabla u)_{\Omega_1} - k\ < v,\ \mathbf{n}_2 \cdot \nabla u >_{\Gamma_{12}} - < v,\ f >_{\Omega_1} = 0 \quad in\ \Omega_2$$

To solve the problem in a monolithic way, the two equations are sum.

$$\left(v, \frac{\partial u}{\partial t}\right)_{\Omega} + k(\nabla v,\ \nabla u)_{\Omega} - k\left(< v,\ \mathbf{n}_1 \cdot \nabla u >_{\Gamma_{12}} + < v,\ \mathbf{n}_2 \cdot \nabla u >_{\Gamma_{12}}\right) - < v,\ f >_{\Omega} = 0$$

where the term: $k\left(< v,\ \mathbf{n}_1 \cdot \nabla u >_{\Gamma_{12}} + < v,\ \mathbf{n}_2 \cdot \nabla u >_{\Gamma_{12}}\right) = 0$ due to the transmission conditions of the problem.

4.1.3 Third task

Obtain the algebraic form of the Dirichlet-to-Neumann operator (Steklov-Poincaré operator) for the left subdomain, departing from given values of u_i^n at time step n, and an interface value u_2^{n+1}.

The time integration algorithm of the 6-noded mesh of the problem is $A \cdot U^{n+1} = B \cdot U^n$:

$$\begin{bmatrix} A_{00} & A_{01} & 0 & 0 & 0 & 0 \\ A_{10} & A_{11} & A_{12} & 0 & 0 & 0 \\ 0 & A_{21} & A_{22} & A_{23} & 0 & 0 \\ 0 & 0 & A_{32} & A_{33} & A_{34} & 0 \\ 0 & 0 & 0 & A_{43} & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} U_0^{n+1} \\ U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \\ U_5^{n+1} \end{bmatrix} = \begin{bmatrix} B_0^n \\ B_1^n \\ B_2^n \\ B_3^n \\ B_4^n \\ B_5^n \end{bmatrix}$$

Considering from the given conditions that both U_0 and U_5 have Dirichlet homogeneous boundary conditions the system is reduced to:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \end{bmatrix} = \begin{bmatrix} B_1^n \\ B_2^n \\ B_3^n \\ B_4^n \end{bmatrix}$$

where node 1 is the only node in the subdomain Ω_1, nodes $3,\ 4$ are in Ω_2 and node 2 is on the boundary.
The system for subdomain Ω_1 is given by:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \end{bmatrix} = \begin{bmatrix} B_1^n \\ B_2^n \end{bmatrix}$$

Then, solving for the first row considering there is information for U_1^n and U_2^{n+1}:

$$A_{11} \cdot U_1^{n+1} = B_1^n - A_{12} \cdot U_2^{n+1}$$

4.1.4 Fourth task

Obtain the algebraic form of the Neumann-to-Dirichlet operator for the right subdomain, departing from given values of u_i^n and an interface value for the fluxes $\phi^{n+1} = k\ \partial_x u^{n+1}$ at the coordinate of node 2.

The second subdomain time integration scheme + fluxes conditions:

$$\begin{bmatrix} A_{22}^{(\Omega_2)} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \\ 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \end{bmatrix} = \begin{bmatrix} B_2^n - A_{21} \cdot U_1^{n+1} - A_{22}^{(\Omega_1)} \cdot U_2^{n+1} \\ B_3^n \\ B_4^n \end{bmatrix}$$

4.1.5 Fifth task

Write down the iterative algorithm for a staggered approach applying Dirichlet boundary conditions at the interface to the left subdomain and Neumann boundary conditions at the interface for the right subdomain.

The staggered approach consists in sending the off-diagonal terms to the RHS of the problem using a prediction of the variable $\tilde{U}_i^{n+1}$.

$$A_X X^{n+1} = F_X + C_X X^n - A_{X^*} \tilde{U}_i^{n+1}$$

And the prediction can be made using a first order approach $\tilde{U}_i^{n+1} = U^n$ or a second order Taylor expansion as $\tilde{U}_i^{n+1} = 2U^n - U^{n-1}$.
At the left side of the subdomain (Ω_2) the equations are:

$$\begin{bmatrix} A_{22}^{(\Omega_2)} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \\ 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \end{bmatrix} = \begin{bmatrix} B_2^n - A_{21} \cdot \tilde{U}_1^{n+1} - A_{22}^{(\Omega_1)} \cdot \tilde{U}_2^{n+1} \\ B_3^n \\ B_4^n \end{bmatrix}$$

At the right side of the subdomain (Ω_1) the equations are:

$$A_{11} \cdot U^{n+1} = B_1^n - A_{12} \cdot \tilde{U}_2^{n+1}$$

4.1.6 Sixth task

Do the same for a substitution and an iteration by subdomains scheme.

To introduce a substitution scheme, once the first equation is used and the value of interest is determined, this value is replaced in the next equation instead of using prediction. At the left side of the subdomain (Ω_2) the equations are:

$$\begin{bmatrix} A_{22}^{(\Omega_2)} & A_{23} & 0 \\ A_{32} & A_{33} & A_{34} \\ 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} U_2^{n+1} \\ U_3^{n+1} \\ U_4^{n+1} \end{bmatrix} = \begin{bmatrix} B_2^n - A_{21} \cdot \tilde{U}_1^{n+1} - A_{22}^{(\Omega_1)} \cdot \tilde{U}_2^{n+1} \\ B_3^n \\ B_4^n \end{bmatrix}$$

At the right side of the subdomain (Ω_1) the equations are:

$$A_{11} \cdot U^{n+1} = B_1^n - A_{12} \cdot U_2^{n+1}$$

4.1.7 Seventh task

Rewrite the algebraic system associated to the left subdomain (Dirichlet boundary conditions at the interface), using Nitsche's method for applying the boundary conditions. How does the condition number of the resulting system of equations vary with the penalty parameter α?

The Nitsche method enforces boundary conditions ($\tilde{u}$) and makes use of a version of the *Penalty method* (α) maintaining the problem symmetric. The weak version of the problem obtained before is re-written as:

$$\left(v, \frac{\partial u}{\partial t}\right)_{\Omega_1} + k\ (\nabla v, \nabla u)_{\Omega_1} - k < v, \mathbf{n}_1 \cdot \nabla u >_{\Gamma_{12}} + \alpha \frac{k}{h} (v, u)_{\Gamma_{12}} - k < \mathbf{n}_1 \cdot \nabla v, u >_{\Gamma_{12}} = < v, f >_{\Omega_1}$$

$$+ \alpha \frac{k}{h} (v, \tilde{u})_{\Gamma_{12}} - k < \mathbf{n} \cdot \nabla v, \tilde{u} >_{\Gamma_{12}}$$

In which the $\tilde{u}$ enforces the boundary conditions. Replacing the values of the solution and the fluxes in the left subdomain:

$$\left(v, \frac{\partial u}{\partial t}\right)_{\Omega_1} - k\ (\nabla v, \nabla u)_{\Omega_1} - k\ [v\ n_1 \cdot (\nabla u)_{x=0.4} - v\ n_2 \cdot (\nabla u)_{x=0}] + \alpha \frac{k}{h} [(v\ u)_{x=0.4} - (v\ u)_{x=0}] - k\ [(n_1 \cdot \nabla v\ u)_{x=0.4}$$

$$- (n_2 \cdot \nabla v\ u)_{x=0}] = < v,\ f >_{\Omega_1} + \alpha\ \frac{k}{h} [(v\ \tilde{u})_{x=0.4} - (v\ \tilde{u})_{x=0}] - k\ \ [(n_1 \cdot \nabla v\ \tilde{u})_{x=0.4} - (n_2 \cdot \nabla v\ \tilde{u})_{x=0}]$$

It is clear from the geometry of the problem (1D), the outgoing n entity has to accomplish $n_1 = -n_2$, moreover it has to be equal to 1, but the expression will be left in terms of n.

$$\left(v, \frac{\partial u}{\partial t}\right)_{\Omega_1} - k\ (\nabla v, \nabla u)_{\Omega_1} - k\ [v\ n_1 \cdot (\nabla u)_{x=0.4} + v\ n_1 \cdot (\nabla u)_{x=0}] + \alpha \frac{k}{h} [(v\ u)_{x=0.4} - (v\ u)_{x=0}] - k\ \ [(n_1 \cdot \nabla v\ u)_{x=0.4}$$

$$+ (n_1 \cdot \nabla v\ u)_{x=0}] = < v,\ f >_{\Omega_1} + \alpha\ \frac{k}{h}\ [(v\ \tilde{u})_{x=0.4} - (v\ \tilde{u})_{x=0}] - k\ [(n_1 \cdot \nabla v\ \tilde{u})_{x=0.4} + (n_1 \cdot \nabla v\ \tilde{u})_{x=0}]$$

Using Galerkin description for the weighting functions equal to those approximating the unknown of interest u, it is possible to write the terms of the previous equation in terms of the matrices K, M.

$$M_{\Omega_1} \cdot \frac{\partial U}{\partial t} - K_{\Omega_1} \cdot U - k\ \left[N_j \frac{\partial N_2}{\partial x} U_{x=0.4} + N_j \frac{\partial N_0}{\partial x} U_{x=0}\right] + \alpha \frac{k}{h}\ [N_j N_2\ U_{x=0.4} + N_j N_0\ U_{x=0}]$$

$$-k\ \left[\frac{\partial N_j}{\partial x} N_2\ U_{x=0.4} + \frac{\partial N_j}{\partial x} N_0\ U_{x=0}\right] = (N_j,\ f)_{\Omega_1} + \alpha \frac{k}{h}\ [N_j N_2\ \tilde{U}_{x=0.4} + N_j N_0\ \tilde{U}_{x=0}]$$

$$-k\ \left[\frac{\partial N_j}{\partial x} N_2\ \tilde{U}_{x=0.4} + \frac{\partial N_j}{\partial x} N_0\ \tilde{U}_{x=0}\right]$$

From where it can be seen that the products between $N_j \cdot N_{0,\ 2}$ will give as a result a vector as well as for their derivatives. Considering $\partial N_{0,\ 2}/\partial x = h/2$ for linear 1D elements and simplifying the expressions.

$$M_{\Omega_1} \cdot \frac{\partial U}{\partial t} - K_{\Omega_1} \cdot U + k\ N_j \left(\left(\frac{\alpha}{h}\ N_2\ - \frac{h}{2}\right)\ U_{x=0.4} + \left(\frac{\alpha}{h}\ N_0\ - \frac{h}{2}\right) U_{x=0}\right) - k \frac{\partial N_j}{\partial x} \cdot [N_2 \cdot U_{x=0.4} + N_0 \cdot U_{x=0}]$$

$$= (N_j,\ f)_{\Omega_1} + \alpha \frac{k}{h} N_j\ [N_2\ \tilde{U}_{x=0.4} + N_0\ \tilde{U}_{x=0}] - k \frac{\partial N_j}{\partial x}\ [N_2\ \tilde{U}_{x=0.4} + N_0\ \tilde{U}_{x=0}]$$

Replacing this expression in an iterative time integration scheme using Nitsche method with Dirichlet boundary conditions $\tilde{U}_2$ is replaced by the Dirichlet boundary conditions from subdomain Ω_2 and imposing global Dirichlet boundary conditions of the problem ($U_0 = 0,\ \forall\ t$). Using theta method for time integration, precisely $BDF1$ scheme.

$$M_{\Omega_1} \cdot \frac{U^{n+1} - U^n}{\Delta t} - K_{\Omega_1} \cdot U^{n+1} + k\ N_j \left(\left(\frac{\alpha}{h}\ N_2\ - \frac{h}{2}\right)\ U^{n+1}_{x=0.4}\right) - k \frac{\partial N_j}{\partial x} N_2 \cdot U^{n+1}_{x=0.4} = \mathbf{f} + \left(\alpha \frac{k}{h}\ N_j - k \frac{\partial N_j}{\partial x}\right)\ N_2 U^n_{x=0.4\ (\Omega_2)}$$

Condition number

As mentioned previously, the *Nitsche method* is better conditioned than the regular *Penalty Method* by enforcing boundary conditions and including terms to make the method symmetric (even though this substract some stability). In the case of a heat transfer equation, like this case, the parameter $\alpha > 2\ c_i$ is sufficient to ensure stability. This parameter c_i depends on the shape of the elements, for a non-stretched element $c_i = \mathbb{O}(1)$. The value of α needs to be devised for different physical problems.

5 Operator splitting technique

5.1 First exercise

Consider the one-dimensional, transient, convection-diffusion equation:

$$\begin{gathered} \frac{\partial u}{\partial t} - k\ \frac{\partial^2 u}{\partial x^2} + a_x\ \frac{\partial u}{\partial x} = f \quad in\ [0,\ 1] \\ u(x = 0, t) = 0 \\ u(x = 1, t) = 0 \\ u(x, t = 0) = 0 \end{gathered}$$

with $k = 1$, $a_x = 1$ and $f = 1$.

5.1.1 First task

Discretize it in space using finite elements (3 elements) and in time (finite differences, $BDF1$). Solve the first step of the problem, writing the solution as a function of the time step size δt.

The transient convection-diffusion strong form is pre-multiplied by a weighting function and then integrated by parts to get the weak form of the problem:

$$\int_\Omega w\ \frac{\partial u}{\partial t}\ d\Omega - \int_\Omega w \left(k\ \frac{\partial^2 u}{\partial x^2}\right)\ d\Omega + \int_\Omega w \left(a_x \cdot \frac{\partial u}{\partial x}\right)\ d\Omega = \int_\Omega wf\ d\Omega$$

The weak form of the problem reads:

$$\int_\Omega w\ \frac{\partial u}{\partial t}\ d\Omega + \int_\Omega \frac{\partial w}{\partial x} \cdot \left(k\ \frac{\partial u}{\partial x}\right)\ d\Omega - \int_\Gamma w\ \frac{\partial u}{\partial x} \cdot \mathbf{n}\ d\Gamma + \int_\Omega w \left(a_x \cdot \frac{\partial u}{\partial x}\right)\ d\Omega = \int_\Omega wf\ d\Omega$$

Now, the description of these functions using finite elements and Galerkin weighting functions, and taking into account that the time will be also discretized:

$$u(x) \approx u^h(x) = \sum_{i=1}^{n_n} N_i(x) \cdot U_i(t)$$

$$w(x) = \sum_{i=1}^{n_n} N_j(x),\ \forall\ W_i$$

Then the discrete matrices are found as:

$$M_{ij} = \int_\Omega N_j \cdot N_i\ d\Omega = \sum_{j=1}^{n_n}\sum_{i=1}^{n_n} N_j(z_{GP}) \cdot N_i(z_{GP})\ w_{GP}$$

$$K_{ij} = \int_\Omega \frac{\partial N_j}{\partial x} \cdot \nu\ \frac{\partial N_i}{\partial x}\ d\Omega = \sum_{j=1}^{n_n}\sum_{i=1}^{n_n} \frac{\partial N_j(z_{GP})}{\partial x} \cdot \nu\ \frac{\partial N_i(z_{GP})}{\partial x}\ w_{GP}$$

$$C_{ij} = \int_\Omega N_j\ a_x \cdot \frac{\partial N_i}{\partial x}\ d\Omega = \sum_{j=1}^{n_n}\sum_{i=1}^{n_n} N_j(z_{GP})\ a_x \cdot \frac{\partial N_i(z_{GP})}{\partial x}\ w_{GP}$$

$$F_i = \int_\Omega N_j \cdot f\ d\Omega = \sum_{j=1}^{n_n} N_j(z_{GP}) \cdot f\ w_{GP}$$

With these definitions, introducing the fact that every boundary has an homogeneous Dirichlet condition, the weak form of the problem can be written as:

$$M \cdot \frac{\Delta u}{\delta t} + K \cdot U + \cdot C \cdot U = F$$

Using backward differences in time scheme:

$$M\frac{\Delta u}{\delta t} + \theta \cdot \left(\ K \cdot U^{n+1} + C \cdot U^{n+1} - F^{n+1}\right) + (1-\theta) \cdot (\ K \cdot U^n + C \cdot U^n - F^n) = 0$$

When using $BDF1$, $\theta = 1$:

$$M\left(U^{n+1} - U^n\right) + \delta t\ \left(\ K \cdot U^{n+1} + C \cdot U^{n+1} - F^{n+1}\right) = 0$$

Considering f is a constant:

$$(M + \delta t\ \ (\ K + C)) \cdot U^{n+1} = \delta t\ F + M \cdot U^n$$

Considering both terms in the LFS are known, a new matrix is introduced $A = (M + \delta t\ \ (\ K + C))$ and a vector $B = \delta t F + M \cdot U^n$. Then, the iterative scheme takes the form:

$$A \cdot U^{n+1} = B$$

The idea now is to solve the first step of the iterative scheme with a 3 element mesh (4 nodes for linear interpolation) and homogeneous Dirichlet Boundary conditions in both ends (Node 0 and Node 3), this reduction of the matrices have to be made have the whole system is calculated and the matrices assembled. Also the given values $\nu = 1$, $a_x = 1$ and $f = 1$ are used.

$$N_i(\xi) = \begin{cases} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{cases}$$

$$\frac{dN_i(\xi)}{dx} = \begin{cases} \frac{-1}{2} \\ \frac{1}{2} \end{cases}$$

$$J = \frac{l^e}{2}$$

The initial condition for $U(x, t=0) = 0$ therefore, in the expression solved is:

$$U^{n+1} = (M + \delta t\ (K + C))^{-1} \cdot \delta t \cdot F$$

$$M = \frac{l^e}{3} \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 2 & 1/2 & 0 \\ 0 & 1/2 & 2 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{bmatrix}$$

$$K = \frac{1}{l^e} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1/2 & 1/2 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}$$

$$F = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/6 \end{bmatrix}$$

Using $l_e = 1/3$ to the expressions:

$$A = M + \delta t\ (K + C) = \begin{bmatrix} 2/9 + 6\delta t & 1/18 - 5/2\delta t \\ 1/18 - 7/2\delta t & 2/9 + 6\delta t \end{bmatrix}$$

The inverse of a 2x2 matrix is:

$$A^{-1} = (M + \delta t\ (K + C))^{-1} = \frac{1}{(15/324) + (13/3)\ \delta t + (109/4)\ \delta t^2} \cdot \begin{bmatrix} 2/9 + 6\delta t & -1/18 + 5/2\delta t \\ -1/18 + 7/2\delta t & 2/9 + 6\delta t \end{bmatrix}$$

We get to the following expression of the first iteration step of the solution:

$$U^{n+1} = A^{-1} \cdot F = \frac{1}{15/108 + 13\ \delta t + (327/4)\ \delta t^2} \cdot \begin{bmatrix} \frac{3}{18} + \frac{17}{2}\delta t \\ \\ \frac{3}{18} + \frac{19}{2}\delta t \end{bmatrix}$$

5.1.2 Second exercise

Solve the same time step by using a first order operator splitting technique.

The idea of the splitting operator techniques is to decouple the equations of the strong form of the problem in order to lower computational cost of solving the coupled problem in an iteration. To this end, the convection-diffusion equation is split as:

$$\mathscr{L}\ u = \mathscr{L}_{\mathbf{a}}\ u + \mathscr{L}_\nu\ u = \mathbf{a}_x \cdot \nabla u - \nu\ \Delta u$$

$$\partial_t u + \mathscr{L}_{\mathbf{a}} u + \mathscr{L}_\nu u = f$$

The first step is to calculate the step forward in the convection problem as:

$$u_a(t^n) = u^n$$

$$\partial_t u_a + \mathscr{L}_{\mathbf{a}} u_a = 0$$

Then we solve for the diffusion problem using the convection problem solution as initial data:

$$u_\nu(t^n) = u^a(t^{n+1})$$

$$\partial_t u_\nu + \mathscr{L}_\nu u_\nu = f$$

Using $BDF1$ scheme, the first step reads:

$$\frac{u_a^{n+1}-u_a^n}{\delta t} + \mathscr{L}_{\mathbf{a}} u_a^{n+1} = 0 \implies u_a^{n+1} = (\mathbf{I} + \delta t\ \mathscr{L}_{\mathbf{a}})^{-1} u^n$$

Then the second step is:

$$\frac{u^{n+1}-u_a^n}{\delta t} + \mathscr{L}_{\nu} u^{n+1} = 0 \implies u^{n+1} = (\mathbf{I} + \delta t\ \mathscr{L}_{\nu})^{-1} (u_a^{n+1} + f\ \delta t)$$

Now into the discrete form of the problem, considering that the operators $\mathscr{L}_{\mathbf{a}}$ and $\mathscr{L}_{\nu}$ will constitute the already obtained matrices C and K respectively, the problem can be written as:

$$U_{\mathbf{a}}^n = U^n = [0;\ 0]^T$$

$$U_{\mathbf{a}}^{n+1} = (\mathbf{M} + \delta t\ \mathbf{C})^{-1} \cdot \mathbf{M}\ U_a^n = [0;\ 0]^T$$

The second step is:

$$U_{\nu}^n = U_{\mathbf{a}}^{n+1} = [0;\ 0]^T$$

$$U^{n+1} = (\mathbf{M} + \delta t\ \mathbf{K})^{-1} \cdot \left(\mathbf{M}\ U_a^{n+1} + F\right) = (\mathbf{M} + \delta t\ \mathbf{K})^{-1} \cdot F$$

Considering the values from previous exercise and taking into account the boundary conditions:

$$M + \delta t \cdot \mathbf{K} = \begin{bmatrix} 2/9 + 6\ \delta t & 1/18 - 3\ \delta t \\ 1/18 - 3\ \delta t & 2/9 + 6\ \delta t \end{bmatrix}$$

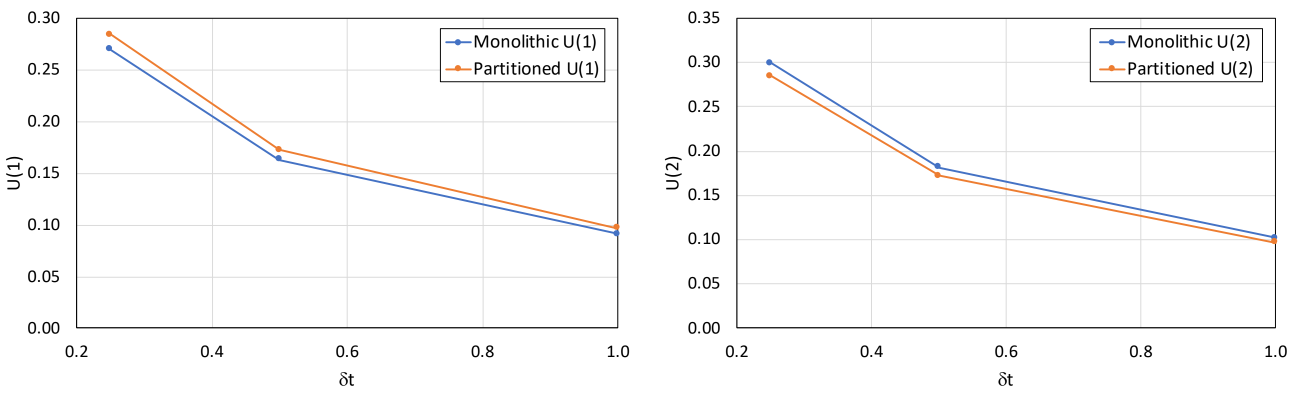
The matrix is inverted and multiplied by F:

$$U^{n+1} = \frac{1}{15/108 + 9\ \delta t + 81\ \delta t^2} \begin{bmatrix} 2/9 + 6\ \delta t & -1/18 + 3\ \delta t \\ -1/18 + 3\ \delta t & 2/9 + 6\ \delta t \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{15/108 + 9\ \delta t + 81\ \delta t^2} \begin{bmatrix} 3/18 + 9\delta t \\ 3/18 + 9\delta t \end{bmatrix}$$

5.1.3 Third task

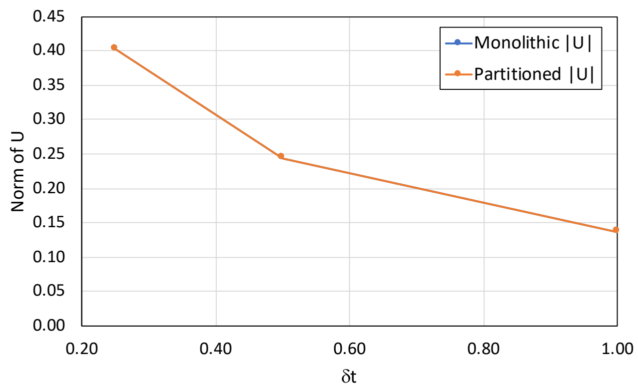
Evaluate the error of the splitting approach with respect to the monolithic approach. Plot the splitting error vs. the time step size for $\delta t = 1$, $\delta t = 0.5$ and $\delta t = 0.25$. Comment on the results.

In Figure 4a, Figure 4b and Figure 5a the results obtained in the previous two tasks are shown. The results show how for the different component there are slight differences according to each of the approaches (Figure 4a and Figure 4b) which cannot be noticed when analysing the norm of the components as in Figure 5a. This is due to the fact that the Partitioned Scheme with the initial guess of the solution made supress the effect of the non-symmetric Convection matrix and both $U(1) = U(2)$ which is not the case for the monolithic case. This can be confirmed by seeing how in Figure 4a and Figure 4b the Monolithic is below and above the Partitioned '*constant*' solution, in this way is not surprising the results found in Figure 5a, as these differences are canceled for the Monolithic when in sum and tend to the mid point where the Partitioned results are. Finally, the error in the partitioned scheme method is plotted against the monolithic scheme ($\varepsilon = abs[(U_{part}(i) - U_{monolithic}(i))/U_{monolithic}(i)]$), in Figure 5b where it is observed how as the time step δt is reduced, the error is diminished.

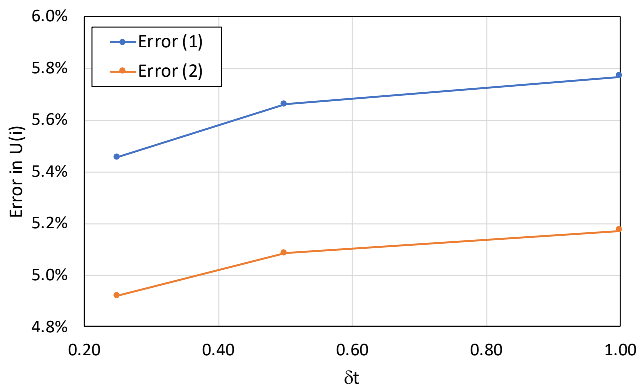


(a) Component 1 (b) Component 2

Figure 4: Monolithic vs. Partitioned time integration schemes.



(a) Norm of both components



(b) Erros in components

Figure 5: Monolithic vs. Partitioned time integration schemes and error in partitioned scheme.

6 Fractional step methods

6.1 First exercise

Consider the fractional step approach for the incompressible Navier-Stokes equations (Yosida scheme):

$$M\ \frac{1}{\delta t}\left(\hat{U}^{n+1}-U^{n}\right)+K\ \hat{U}^{n+1}=f-G\ \tilde{P}^{n+1}$$

$$D\ M^{-1}\ G\ P^{n+1}=\frac{1}{\delta t}D\ \hat{U}^{n+1}-D\ M^{-1}\ G\ \tilde{P}^{n+1}$$

$$M\ \frac{1}{\delta t}\left(U^{n+1}-\hat{U}^{n+1}\right)+\alpha K\ \left(U^{n+1}-\hat{U}^{n+1}\right)+G\ \left(P^{n+1}-\tilde{P}^{n+1}\right)=0$$

6.1.1 First task

Which is the optimal value for α parameter?
Analysing the problem of interest, it is seen that the problem involves an unsteady version of the Navier-Stokes equation, which will be fully implicitly integrated with $BDF1$:

$$M\ \frac{1}{\delta t}\left(U^{n+1}-U^{n}\right)+K\ U^{n+1}+G\ P^{n+1}=f$$

If the *Yosida scheme* recovers the implicit time integration, the method would be optimal. To this end, considering we want to enforce no second step to be performed (as would be performed in a one-step implicit method) the first and third equations are used:

$$\frac{M}{\delta t}\hat{U}^{n+1}-\frac{M}{\delta t}U^{n}+K\ \hat{U}^{n+1}=f-G\ \tilde{P}^{n+1}$$

$$\frac{M}{\delta t}U^{n+1}-\frac{M}{\delta t}\hat{U}^{n+1}+\alpha K\ U^{n+1}-\alpha K\ \hat{U}^{n+1}+GP^{n+1}-G\tilde{P}^{n+1}=0$$

Now we sum both equations:

$$\frac{M}{\delta t}\hat{U}^{n+1}-\frac{M}{\delta t}\hat{U}^{n+1}+\frac{M}{\delta t}U^{n+1}-\frac{M}{\delta t}U^{n}+\alpha K\ U^{n+1}+K\ \hat{U}^{n+1}-\alpha K\ \hat{U}^{n+1}+GP^{n+1}-G\tilde{P}^{n+1}=f-G\ \tilde{P}^{n+1}$$

$$\frac{M}{\delta t}(U^{n+1}-U^{n})+\alpha\ K\ U^{n+1}+K\ \tilde{U}^{n+1}\ (1-\alpha)+G\ P^{n+1}=f$$

It is clearly seen that for $\alpha=1$ the above equation is equal to that in the fully implicitly integration with $BDF1$.

6.1.2 Second task

What is the source of error of the scheme?
The source of error of the scheme is the calculation of the first inverse used. In the *Yosida scheme* the fractional step is introduced by converting the original system into a LU matrix as:

$$\begin{bmatrix} K & G \\ D & 0 \end{bmatrix} = \begin{bmatrix} K & 0 \\ D & -D\ H_1\ G \end{bmatrix} \begin{bmatrix} I & H_2\ G \\ 0 & I \end{bmatrix} = \begin{bmatrix} K & K\ H_2\ G \\ D & D\ (H_2 - H_1)\ G \end{bmatrix}$$

From these expressions, the *Yosida scheme* has $H_1 = \delta t M^{-1}$ and $H_2 = K^{-1}$, and this is the source of the error which can clearly be reduced for $H_1 = H_2 = \delta t\ [I - \delta t\ M^{-1}(\nu L + N)]\ M$.

It is pointed out here that in these exercises $K = M/\delta t + \nu L + N,\ M = (N_i, N_j,\ L = (\partial N_i, \partial N_j)$ and $N = (N_i, \mathbf{a} \cdot \partial N_j)$.

7 ALE formulations

7.1 First exercise

Given the spatial description of a property:

$$\gamma(x, y, z, t) = [2x, ye^t, z]^T$$

The equations of movement:

$$x = \mathbf{X}e^t$$
$$y = \mathbf{Y} + e^t - 1$$
$$z = \mathbf{Z}$$

And the equations of movement of the mesh:

$$x_m = \mathscr{X} + \alpha t$$
$$y_m = \mathscr{Y} - \beta t$$
$$z_m = \mathscr{Z}$$

7.1.1 First task

Obtain the description of the property in terms of the ALE coordinates ($\mathscr{X}, \mathscr{Y}, \mathscr{Z}$).
The idea is to obtain the description of the property $\gamma(x, y, z, t)$ as $\gamma(\underline{x}_m(\underline{\mathscr{X}}), t)$. This is to say:

$$\gamma_{ALE}(\underline{\mathscr{X}}, t) = \left[2\ (\mathscr{X} + \alpha t), (\mathscr{Y} - \beta t)\ e^t, \mathscr{Z}\right]$$

7.1.2 Second task

Compute the velocity of the particles and the mesh velocity.
The velocity of the particles can be obtained by deriving $\partial \underline{x}(\underline{\mathbf{X}})/\partial t$:

$$v_p(\mathbf{X}) = \frac{\partial x(\mathbf{X})}{\partial t} = [\mathbf{X}\ e^t, e^t, 0]$$

The velocity of the mesh is obtained as $\partial \underline{x}_m(\underline{\mathscr{X}})/\partial t$:

$$v_m(\underline{\mathscr{X}}) = \frac{\partial \underline{x}_m(\underline{\mathscr{X}})}{\partial t} = [\alpha, -\beta, 0]$$

7.1.3 Third task

Compute the ALE description of the material temporal derivative of γ.
The material temporal derivative of a property has to take into account the partial time derivative $\partial\gamma/\partial t$ and the associated convection fluxes $v \cdot \nabla\gamma\ (\underline{x}, t)$. Therefore the derivative of interest is:

$$\frac{d}{dt}\gamma_{ALE}(\underline{\mathscr{X}}, t) = \frac{\partial\gamma(\underline{x}, t)}{\partial t} + (v_p - v_m)^T \cdot \nabla\gamma(\underline{x}, t) = \begin{bmatrix} 2\ \alpha \\ (\mathscr{Y} - \beta\ (1+t))\ e^t \\ 0 \end{bmatrix}^T + \begin{bmatrix} \mathbf{X}\ e^t - \alpha \\ e^t + \beta \\ 0 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{d}{dt}\gamma_{ALE}(\underline{\mathscr{X}}, t) = \begin{bmatrix} 2\ \mathbf{X}\ e^t & (\mathscr{Y} - \beta\ t + e^t)\ e^t & 0 \end{bmatrix}$$

To fully express the above expression in terms of the ALE coordinates, introduce the relationship $\mathbf{X}e^t = x = \mathscr{X} + \alpha\ t$.

$$\frac{d}{dt}\gamma_{ALE}(\underline{\mathscr{X}}, t) = \begin{bmatrix} 2\ (\mathscr{X} + \alpha\ t) & (\mathscr{Y} - \beta\ t + e^t)\ e^t & 0 \end{bmatrix}$$

7.2 Second exercise

Write down the ALE form of the incompressible Navier-Stokes equations. Where (in time and space) is each of the terms of the equation evaluated? How are temporal derivatives computed?
The *incompressible Navier-Stokes* equation, consider the conservation of momentum, the Newtonian fluid stress tensor and, of course, the mass conservation of an incompressible fluid. To this end, first the momentum conservation in ALE form is:

$$\frac{\underline{u}_{ALE}(\partial\underline{\mathscr{X}}, t)}{\partial t} + \underline{c} \cdot \underline{\nabla}\ \underline{u}(\underline{x}, t) = \nabla \cdot \underline{\underline{\sigma}}(\underline{x}, t) + \underline{b}(\underline{x}, t)\ \rho(\underline{x}, t) \tag{21}$$

The definition of the Newtonian fluid stress tensor $\underline{\underline{\sigma}}(\underline{x}, t)$ is:

$$\underline{\underline{\sigma}}(\underline{x}, t) = -p(\underline{x}, t)\ \underline{\underline{\mathbf{I}}} + 2\mu\ \underline{\nabla}^s\underline{u}(\underline{x}, t)$$

Considering this definition in Eq. (21) reads:

$$\frac{\underline{u}_{ALE}(\partial\underline{\mathscr{X}}, t)}{\partial t} + \underline{c} \cdot \underline{\nabla}\ \underline{u}(\underline{x}, t) + \nabla p(\underline{x}, t) - \mu\ \underline{\nabla}^2\underline{u}(\underline{x}, t) = \underline{b}(\underline{x}, t)\ \rho(\underline{x}, t)$$

where $\underline{c} = v_p - v_{mesh}$
Then the mass conservation in ALE description is:

$$\frac{\partial\rho_{ALE}(\mathscr{X}, t)}{\partial t} + \underline{c} \cdot \nabla\rho(\underline{x}, t) = -\rho\ \underline{\nabla} \cdot \underline{u}(\underline{x}, t)$$

Considering the fluid incompressible, the first two terms in the above equation constitutes a material derivative of ρ and for mass conservation are zero, therefore the final expression is:

$$\underline{\nabla} \cdot \underline{u}(\underline{x}, t) = 0$$

The final expression of an *incompressible Navier-Stokes* in ALE description are described in Eq. (22) and Eq. (23).

$$\frac{\underline{u}_{ALE}(\partial\underline{\mathscr{X}}, t)}{\partial t} + \underline{c} \cdot \underline{\nabla}\ \underline{u}(\underline{x}, t) + \nabla p(\underline{x}, t) - \mu\ \underline{\nabla}^2\underline{u}(\underline{x}, t) = \underline{b}(\underline{x}, t)\ \rho(\underline{x}, t) \tag{22}$$

$$\underline{\nabla} \cdot \underline{u}(\underline{x}, t) = 0 \tag{23}$$

7.3 Third exercise

Do a bibliographical research on existing methods for the definition of the mesh movement in ALE formulations (Poisson problem, Elasticity problem, etc.). Describe the main advantages of each of these methods.

In ALE formulations neither the material ($R_{\underline{\mathbf{X}}}$) nor the spatial ($R_{\underline{x}}$) configurations are taken as reference, thus a third domain is needed, the referential configuration ($R_{\underline{\mathscr{X}}}$). This new domain relates to the previously mentioned two using $\mathbf{\Psi}$ and $\mathbf{\Phi}$ and the particle motion can be recovered as: $\varphi = \mathbf{\Phi}\ o\ \mathbf{\Psi}^{-1}$ clearly stating there is a dependency between these maps, as seen in Figure 6 [1]. Both the material and the mesh move with respect to a fixed point.

Thus, the corresponding material and mesh velocities are defined, deriving with respect to time the equations of material motion and mesh motion respectively defining the convective velocity $\underline{c}$, the relative velocity between the material and the mesh. The ALE formulation enables to conveniently deal with moving boundaries, free surfaces and large deformations, and interface contact problems, [2].

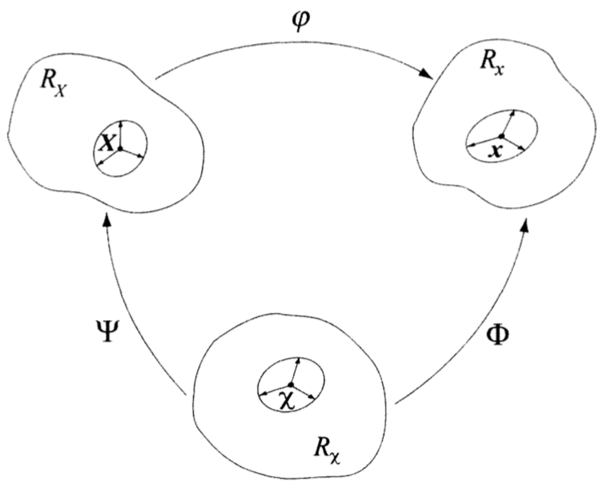


Figure 6: Arbitrary Lagrangian-Eulerian description scheme, Donea [1].

Given this description one of the key points in ALE formulations is the automatic mesh displacement algorithm. To this end, different approaches for various natures were developed, and are explained in the following lines.

Mesh dynamics

The methods to handle this can be classified into fixed and moving grid approaches Figure 7. Fixed-grid methods can be employed together with fictitious domain, level set, or Chimera techniques. Moving mesh approaches can be realized either by simple interpolation techniques or by more sophisticated pseudo-structure techniques of varying complexity.

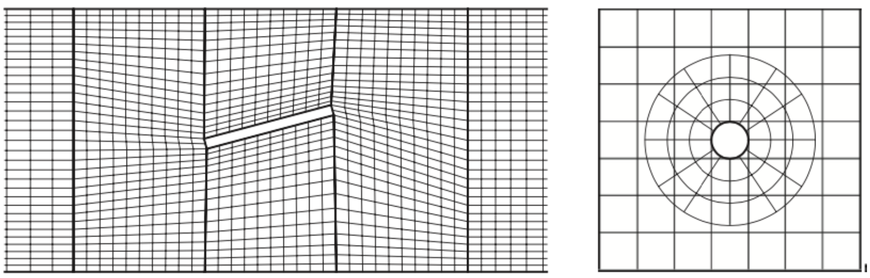


Figure 7: Examples of moving (left) and fixed (right) grids, Benson [3].

Poisson problem

One of the typical methods for generating moving meshes is based on *elliptic approaches* of the strong form of the elliptic Poisson equation. In this way, the method changes the position of nodes without modifying the topology of the mesh. As an advantage, the method is relatively inexpensive to compute and allows to adapt the mesh refinement in high gradients generating orthogonal meshes therefore suppressing the inversion errors. As a drawback it does not guarantee an improvement in the mesh quality for all cases.

Algebraic interpolation

This family of methods include interesting and versatile approaches for different nature of problems. One of these methods are the algebraic grid generation, that are faster, have direct control over the node location loosing control over the grid smoothness. Another important drawback is the imposition of restrictions on the mesh topology, when two curves are discretized with the same number of elements.

Mesh adaptitivity

This method is of special interest for highly variable solutions. The idea is to control the density of a mesh by adapting to geometric, analysis, or predefined properties. These adaptations can be applied with fixed intervals or through sizing functions. Some of the typical strategies for defining sizing functions are based on the analysis of previous solutions.

8 Fluid-Structure interaction

8.1 First exercise

Describe the added mass effect problem for fluid-structure interaction problems. When does it appear, what kind of problems suffer from it? What are the main methods for dealing with it?

The effect appears when using partitioned approaches (explicit and iterating within a time step) the convergence is not guaranteed for incompressible flows, when the fluid's density is similar to the solid's density (biomechanics, body tissues vs. water). The methods for overcoming this issue are the *Aitken relaxation method*, the *steepest descend methods* and applying *Robin-Robin boundary conditions.*

8.2 Second exercise

Consider the iteration by subdomain scheme for the heat transfer problem described in problem 1. Apply 2 iterations of the AITKEN relaxation scheme to it.

The one dimensional heat transfer problems is given in Eq. (24).

$$\begin{cases} \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f & in\ [0,\ 1] \\ u(x=0,t) = 0 = \tilde{U}_1 \\ u(x=1,t) = 0 = \tilde{U}_2 \\ u(x,t=0) = 0 \\ k = f = 1 \end{cases} \tag{24}$$

The domain is subdivided into two parts. In subdomain 1 reads:

$$\begin{cases} \frac{\partial u_1^{(n+1)(i)}}{\partial t} - k_1\ \frac{\partial^2 u_1^{(n+1)(i)}}{\partial x^2} = f & in\ \Omega_1 \\ u_1^{(n+1)(i)} = \tilde{U}_1 & in\ \Gamma_1 \\ k_1\ \frac{\partial u_1^{(n+1)(i)}}{\partial n} = k_2\ \frac{\partial u_2^{(n+1)(i-1)}}{\partial n} & in\ \Gamma_{12} \end{cases}$$

Subdomain 2 reads:

$$\begin{cases} \frac{\partial u_2^{(n+1)(i)}}{\partial t} - k_2\ \frac{\partial^2 u_2^{(n+1)(i)}}{\partial x^2} = f & in\ \Omega_2 \\ u_2^{(n+1)(i)} = \tilde{U}_2 & in\ \Gamma_2 \\ u_2^{(n+1)(i)} = u_1^{(n+1)(l)} & in\ \Gamma_{12} \end{cases}$$

where $l = i-1$ is Jacobi scheme and $l = i$ is for Gauss-Seidel.
The *Aitken relaxation method* needs three iterations for the calculus of the relaxation parameter, for this reason the first iteration adopted is $i = 2$ and the second will be $i = 3$.

<u>First iteration</u> ($i = 2$):

$$\begin{cases} \frac{\partial u_1^{(n+1)(2)}}{\partial t} - k_1\ \frac{\partial^2 u_1^{(n+1)(2)}}{\partial x^2} = f & in\ \Omega_1 \\ u_1^{(n+1)(2)} = \tilde{U}_1 & in\ \Gamma_1 \\ k_1\ \frac{\partial u_1^{(n+1)(2)}}{\partial n} = k_2\ \frac{\partial u_2^{(n+1)(1)}}{\partial n} & in\ \Gamma_{12} \end{cases}$$

Subdomain 2 reads, considering Jacobi Scheme:

$$\begin{cases} \frac{\partial u_2^{(n+1)(2)}}{\partial t} - k_2 \frac{\partial^2 u_2^{(n+1)(2)}}{\partial x^2} = f & in\ \Omega_2 \\ u_2^{(n+1)(2)} = \tilde{U}_2 & in\ \Gamma_2 \\ u_2^{(n+1)(2)} = u_1^{(n+1)(1)} + \omega \left(u_1^{(n+1)(2)} - u_2^{(n+1)(1)} \right) & in\ \Gamma_{12} \\ \omega = \frac{u_2^n - u_2^{(n+1)(1)}}{u_2^n - u_2^{(n+1)(1)} + u_1^{(n+1)(2)} - u_1^{(n+1)(1)}} \end{cases}$$

<u>Second iteration</u> ($i = 3$):

$$\begin{cases} \frac{\partial u_1^{(n+1)(3)}}{\partial t} - k_1 \frac{\partial^2 u_1^{(n+1)(3)}}{\partial x^2} = f & in\ \Omega_1 \\ u_1^{(n+1)(3)} = \tilde{U}_1 & in\ \Gamma_1 \\ k_1 \frac{\partial u_1^{(n+1)(3)}}{\partial n} = k_2 \frac{\partial u_2^{(n+1)(2)}}{\partial n} & in\ \Gamma_{12} \end{cases}$$

Subdomain 2 reads, considering Jacobi Scheme:

$$\begin{cases} \frac{\partial u_2^{(n+1)(3)}}{\partial t} - k_2 \frac{\partial^2 u_2^{(n+1)(3)}}{\partial x^2} = f & in\ \Omega_2 \\ u_2^{(n+1)(3)} = \tilde{U}_2 & in\ \Gamma_2 \\ u_2^{(n+1)(3)} = u_1^{(n+1)(2)} + \omega \left(u_1^{(n+1)(3)} - u_2^{(n+1)(2)} \right) & in\ \Gamma_{12} \\ \omega = \frac{u_2^{(n+1)(1)} - u_2^{(n+1)(2)}}{u_2^{(n+1)(1)} - u_2^{(n+1)(2)} + u_1^{(n+1)(3)} - u_1^{(n+1)(2)}} \end{cases}$$

8.3 Third exercise

Consider the monolithic (1 domain), transient (BDF1), finite element (linear elements, $h = 1/4$) approximation of the heat transfer equation in problem 1. Enforce the Dirichlet boundary conditions in $x = 0$ and $x = 1$ by using Lagrange multipliers. What is the form of the discrete system? What is the condition number of the resulting matrix?

From the results obtained in Section 5.1.1, the discrete matrices of interest can be written as:

$$K_G = \frac{1}{l^e} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$M_G = \frac{l^e}{3} \begin{bmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 1/2 & 2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

The imposition of Dirichlet boundary conditions using Lagrange multipliers has the associated matrix:

$$L_\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then the system is:

$$\begin{bmatrix} \frac{M}{\delta t} + K & L_\lambda^T \\ L_\lambda & 0 \end{bmatrix} \begin{bmatrix} U^{n+1} \\ \lambda \end{bmatrix} = \begin{bmatrix} f + \frac{M}{\delta t} U^n \\ \tilde{U}_{\Omega_i} \end{bmatrix}$$

The condition number of the resulting matrix is using $k_A = ||A|| \cdot ||A||^{-1}$, the result for $\delta t = 0.1$ is $k_A = 58.5$ and $\delta t = 1.0$ is $k_A = 38.3$ (using MATLAB).

8.4 Fourth exercise

Consider the monolithic (1 domain), transient (BDF1), finite element (linear elements, $h = 1/4$) approximation of the heat transfer equation in problem 1. Suppose that a level set function ($\Psi = 0$ at $x = 0.4$) divides the domain into a high thermal conductivity ($k = 100$) subdomain ($x \in [0, 0.4]$) and a low thermal conductivity ($k = 1$) subdomain ($x \in [0.4, 1.0]$). Build the system matrix for this problem. Take into account the need for subintegrating the element cut by the level set function.

It is clear from the data given that the elements have a length $h = 0.25$ and the first subdomain ends at $\Omega_1 = [0, 0.4]$ therefore the domain is split through the second element. From the previous exercise, the difference can only be noticed when analysing the stiffness matrix of element no. 2. Therefore, the integration of the matrix, needs to be recalculated as (considering the integration in isoparametric coordinates goes from $[-1.0; 1.0]$ and the element goes from $[0.25; 0.4] \cup [0.4; 0.5]$, then in isoparametric coordinates the domain spliting is $[-1.0; 0.2] \cup [0.2; 1.0]$).

$$K_{22} = \int_{\Omega_2} \frac{\partial N_i(\xi)}{\partial \xi} \cdot \frac{\partial N_j(\xi)}{\partial \xi} \; J \; d\xi = \int_{-1.0}^{0.2} \frac{\partial N_i(\xi)}{\partial \xi} \cdot \frac{\partial N_j(\xi)}{\partial \xi} \; J \; d\xi + \int_{0.2}^{1.0} \frac{\partial N_i(\xi)}{\partial \xi} \cdot \frac{\partial N_j(\xi)}{\partial \xi} \; J \; d\xi$$

$$K_{22} = k_1 \frac{2}{l^e} \begin{bmatrix} 3/10 & -3/10 \\ -3/10 & 3/10 \end{bmatrix} + k_2 \frac{2}{l^e} \begin{bmatrix} 1/5 & -1/5 \\ -1/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 241.6 & -241.6 \\ -241.6 & 241.6 \end{bmatrix}$$

Assembling considering that element 1 has $k = 100$ and element 3 has $k = 1$:

$$K_G = \begin{bmatrix} 400 & -400 & 0 & 0 & 0 \\ -400 & 641.6 & -241.6 & 0 & 0 \\ 0 & -241.6 & 245.6 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}$$

Considering the mass matrix does not modify its expression in the splitting of the domain of the element:

$$M_G = \frac{l^e}{3} \begin{bmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 1/2 & 2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

Then the system for the problem reads:

$$\left(\frac{M_G}{\delta t} + K_G \right) \; U^{n+1} = f + \frac{M_G}{\delta} \; U^n$$

That can be solved applying Lagrange multipliers or direct reduction of the matrices and unknowns.

References

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[3] D. Benson and M. Souli, *Arbitrary Lagrangian-Eulerian and fluid-structure interaction: numerical simulation.* Wiley, 2010.