Coupled Problems
Homework
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1. Transmission conditions

Deflection $v(x)$ of an Euler-Bernoulli beam:

$$EI \frac{d^4 v}{dx^4} = f$$

Boundary conditions: Dirichlet : $\left.\begin{matrix} x=0 \\ x=L \end{matrix}\right\}$ clamped

Principle of Virtual Work (PTV):

$$EI \int_0^L \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2} = \int_0^L \delta v \, f$$

$$\delta v(0) = \delta v(L) = 0 \, ; \quad \frac{d \delta v}{dx}(0) = \frac{d \delta v}{dx}(L) = 0$$

a) Space of functions for v and δv:

$$\delta v \in H^2(\Omega) \Rightarrow \delta v\big|_{\partial\Omega} = \frac{d\delta v}{dx}\bigg|_{\partial\Omega} = 0$$

$$v \in H^2(\Omega) \Rightarrow v\big|_{\partial\Omega} = \frac{dv}{dx}\bigg|_{\partial\Omega} = 0$$

$H^2(\Omega)$ is defined as the space of functions which belong to Ω such that the first-order and second-order derivatives of the function are square integrably:

$$H^2(\Omega) = \left\{ v : \Omega \to \mathbb{R} \;\middle|\; \int_\Omega v^2 < \infty, \int_\Omega |\nabla v|^2 < \infty, \int_\Omega |\Delta v|^2 < \infty \right\}$$

Notice that in the integral that appears on the left hand side of the weak form, the second-order derivatives of δv and v have to be square integrable.
Moreover, the function f must ensure $\int_0^L \delta v < \infty$.

This means that f must belong to the dual space of δv.

b) If $[0,L] = [0,P] \cup (P,L]$, transmission condition at P implied by regularity requirements.
As it said previously, $v \in H^2(\Omega)$, therefore, v have to be continous, as its derivative, in all the domain

Thanks to Sobolev's inequality that states for $p > 1$ a function in W_p^k is C^m.

$$k - m > n/p$$

Here, by definition $\Rightarrow H^2(\Omega) = W_2^2(\Omega)$

So, $k = p = 2$. $n = 1$ which is the dimension of the domain. So

$$2 - m > \frac{1}{2} \; ; \quad m < 1.5 \Rightarrow m = 1$$

Meaning that $[u \in C^1] \Rightarrow$ functions whose derivative is continous.

It is stated by regularity requirements the two transmission conditions

$$\# \quad [[v]]_P = 0$$

$$\# \quad \left[\left[\frac{dv}{dx}\right]\right]_P = 0$$

$\Big\}$ strong continuity

where, $[[\;]]_P$ computes the jump produced at P.

$$[[v]]_P = \lim_{\varepsilon \to 0} \left[v(P+\varepsilon) - v(P-\varepsilon) \right]$$

$$\left[\left[\frac{dv}{dx}\right]\right]_P = \lim_{\varepsilon \to 0} \left[\frac{dv}{dx}(P+\varepsilon) - \frac{dv}{dx}(P-\varepsilon) \right]$$

c) Obtain the transmission conditions at P, imposing in the PTV that the integral is additive.

* Discretizating and obtaing the weak form.

$$\int_0^L EI\,\delta v \frac{d^4 v}{dx^4} = \int_0^L \delta v\, f\, dx =$$

$$= \int_0^P EI\,\delta v \frac{d^4 v}{dx^4} + \int_P^L EI\,\delta v \frac{d^4 v}{dx^4}\, dx$$

* Integrating by parts.

$$\int_0^P EI\,\delta v \frac{d^4 v}{dx^4}\, dx + \int_L^P EI\,\delta v \frac{d^4 v}{dx^4}\, dx =$$

$$= -\int_0^P EI \frac{d\delta v}{dx} \frac{d^3 v}{dx^3}\, dx + \left[EI\,\delta v \frac{d^3 v}{dx^3} \right]_0^P - \int_P^L EI \frac{d\delta v}{dx} \frac{d^3 v}{dx^3}\, dx +$$

$$+ \left[EI\,\delta v \frac{d^3 v}{dx^3} \right]_P^L = \int_0^P EI \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2}\, dx +$$

$$+ \left[EI\,\delta v \frac{d^3 v}{dx^3} \right]_0^P - \left[EI \frac{d\delta v}{dx} \frac{d^2 v}{dx^2} \right]_0^P + \int_P^L EI \frac{d^2 \delta v}{dx^2} \frac{d^2 v}{dx^2}\, dx +$$

$$+ \left[EI\,\delta v \frac{d^3 v}{dx^3} \right]_P^L - \left[EI \frac{d\delta v}{dx} \frac{d^2 v}{dx^2} \right]_P^L$$

* Virtual displacement function:

$$\delta v(0) = \delta v(L) = 0, \quad \frac{d\delta v}{dx}(0) = \frac{d\delta v}{dx}(L) = 0$$

$$\int_0^L \delta v\, f\, dx = \int_0^P EI \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx + \int_P^L EI \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx$$

$$+ EI\,\delta v(P^-)\frac{d^3 v(P^-)}{dx^3} - EI\frac{d\delta v(P^-)}{dx}\frac{d^2 v(P^-)}{dx^2} -$$

$$- EI\,\delta v(P^+)\frac{d^3 v(P^+)}{dx^3} + EI\frac{d\delta v(P^+)}{dx}\frac{d^2 v(P^+)}{dx^2}$$

* Imposing that the integral is additive in PTV

$$\int_0^L \delta v\, f\, dx = \int_0^L EI \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx = \int_0^P EI \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx +$$

$$+ \int_P^L EI \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx$$

* $\int_0^L \delta v\, f\, dx = \int_0^L \delta v\, f\, dx \Rightarrow$

$$\int_0^P EI \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx + \int_P^L EI \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx =$$

$$= \int_0^P EI \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx + \int_P^L EI \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\,dx +$$

$$+ EI\,\delta v(P^-)\frac{d^3(P^-)}{dx^3} - EI\frac{d\delta v(P^-)}{dx}\frac{d^2 v(P^-)}{dx^2} -$$

$$- EI\,\delta v(P^+)\frac{d^3 v(P^+)}{dx^3} + EI\frac{d\delta v(P^+)}{dx}\frac{d^2 v(P^+)}{dx}$$

Finally:

$$EI\,\delta v(P^-)\,\frac{d^3 v(P^-)}{dx^3} - EI\,\frac{d\delta v(P^-)}{dx}\,\frac{d^2 v(P^-)}{dx} -$$

$$- EI\,\delta v(P^+)\,\frac{d^3 v(P^+)}{dx^3} + EI\,\frac{d\delta v(P^+)}{dx}\,\frac{d^2 v(P^+)}{dx^2} = 0$$

$$\begin{cases} \delta v(P^-) = \delta v(P^+) \\ EI\,\dfrac{d^3 v(P^-)}{dx} - EI\,\dfrac{d^3 v(P^+)}{dx^3} = 0 \end{cases}$$

$$\begin{cases} \dfrac{d\delta v}{dx}(P^-) = \dfrac{d\delta v}{dx}(P^+) \\ -EI\,\dfrac{d^2 v(P^-)}{dx^2} + EI\,\dfrac{d^2 v(P^+)}{dx^2} = 0 \end{cases}$$

Therefore, the 4 transmission conditions are:

$$[[\delta v]] = 0 \quad ; \quad \left[\left[\frac{d\delta v}{dx} = 0\right]\right]$$

$$\left[\left[EI\,\frac{d^2 v}{dx^2}\right]\right] = 0 \quad ; \quad EI\left[\left[\frac{d^3 v}{dx^3}\right]\right] = 0$$

2. Maxwell problem:

$$\nu \nabla \times \nabla \times u = f \quad \text{in } \Omega$$
$$\nabla \cdot u = 0 \quad \text{in } \Omega$$
$$n \times u = 0 \quad \text{on } \partial\Omega$$

a) * Taking:

$$\nabla \times (\nu \nabla \times u) = f \quad \text{in } \Omega$$

* Multiplying by a test function $\delta u \in \mathbb{R}^3$:

$w \times n = 0$ on $\partial\Omega$, in order to apply Dirichlet BC:

$$\int_\Omega w \cdot \nabla \times (\nu \nabla \times u)\, d\Omega = \int_\Omega w \cdot f \, d\Omega$$

* Using: $u (\nabla \times v) = v \cdot \nabla \times u - \nabla \cdot (u \times v)$

$$\int_\Omega w \cdot \nabla \times (\nu \nabla \times u)\, d\Omega = \int_\Omega \nabla \times w \cdot (\nu \nabla \times u)\, d\Omega -$$

$$- \int \nabla \cdot (w \times (\nu \nabla \times u))\, d\Omega = \int_\Omega \nabla \times w \cdot (\nu \nabla \times u)\, d\Omega +$$

$$+ \int_{\partial\Omega} w \times n \cdot (\nu \nabla \times u)\, d\Gamma = \left[\int_\Omega \nabla \times \delta u \cdot (\nu \nabla \times u)\, d\Omega = \right.$$

$$\left. = \int_\Omega w \, f \, d\Omega \right]$$

$\nabla \times w \cdot (\nu) \nabla \times u)$ must be in $L_1(\Omega)$ to assure that the integral is bounded. Being the two terms multiplied in $L_2(\Omega)$

To assure the weak form is symmetric:

$$u \in H^0(\text{curl}, \Omega)$$

b)

The regularity requirement states that:

$$\int_\Omega |\underline{\nabla} \times \underline{u}|^2 \, d\Omega < \infty$$

The transmission condition that satisfies is

$$\left[[[n \times u]]_\Gamma = 0 \right]$$

Being the jump operator:

$$[[n \times u]]_\Gamma = \lim_{\varepsilon \to 0} \left[n \times u(x_0 + \varepsilon) - n \times u(x_0 - \varepsilon) \right]$$

If $\underline{n} \times \underline{u}$ is discontinuous at Γ, the $\nabla \times u$ is defined as:

\#

$$\nabla \times u = \begin{cases} \nabla \times u & x \in \Omega_1 \\ \frac{1}{2\varepsilon}\left(n \times u(x_0+\varepsilon) - n \times u(x_0-\varepsilon)\right) & x_0+\varepsilon < x < x_0-\varepsilon \\ \nabla u & x \in \Omega_2 \end{cases}$$

\# Therefore the square integral of $\nabla \times u$ is evaluated as:

$$\int_{\Omega} (\nabla \times u)^2 = \int_{\Omega_1} (\nabla \times u)^2 + \int_{x_0+\varepsilon}^{x_0-\varepsilon} \left(\frac{1}{2\varepsilon} \left[n \times u(x_0+\varepsilon) - n \times u(x_0-\varepsilon) \right] \right)^2 + \int_{\Omega_2} (\nabla \times u)^2$$

\# If $\varepsilon \to 0 \Rightarrow \int_{\Omega} (\nabla \times u)^2 = \infty$

\# Therefore, if $n \times n$ would be discontinous, then $\nabla \times u$ would not be square integrable, so $u \notin H(curl, \Omega)$

c) * Considering that the integrals of the weak form are additive

* Domain 1:

$$\int_{\Omega_1} (\nabla \times w) \cdot (\nu \nabla \times u) - \int_{\partial\Omega_1 \cap \partial\Omega} w \cdot (\nu \nabla \times u \times n_1) -$$

$$- \int_{\partial\Omega_1 \cap \Gamma} w \cdot (\nu \nabla \times u \times n_1) = \int_{\Omega_1} w f$$

* Domain 2:

$$\int_{\Omega_2} (\nabla \times w) \cdot (\nu \nabla \times u) - \int_{\partial\Omega_2 \cap \partial\Omega} w \cdot (\nu \nabla \times u \times n_2) -$$

$$- \int_{\partial\Omega_2 \cap \Gamma} w \cdot (\nu \nabla \times u \times n_2) = \int_{\Omega_2} w f$$

* Adding both equations and comparing with the weak form of the full domain Ω

$$\int_{\Gamma} w \cdot \nu (\nabla \times u \times n_1 + \nabla \times u \times n_2) = 0$$

* The transmission condition is

$$\left[[[\nu (\nabla \times u \times n)]]_{\Gamma} = 0 \right]$$

B. Navier equations:

$$-2\mu \nabla\cdot(\varepsilon(u)) - \lambda \nabla(\nabla\cdot u) = \rho b$$

$$-\mu \Delta u - (\lambda+\mu)\nabla(\nabla\cdot u) = \rho b$$

$$\mu \nabla\times(\nabla\times u) - (\lambda+2\mu)\nabla(\nabla\cdot u) = \rho b$$

a) Variational form

* 1st equation:

$$\int_\Omega w\cdot\left(-2\mu\nabla\cdot(\varepsilon(u)) - \lambda\nabla(\nabla\cdot u)\right)d\Omega = \int w\,\rho b\,d\Omega$$

* Integrating by parts

$$\int_\Omega \nabla w : 2\mu(\varepsilon(u))\,d\Omega - \int_{\partial\Omega} w\cdot 2\mu\,\varepsilon(u)\cdot \underline{n}\,d\Gamma +$$

$$+\int_\Omega \nabla w\cdot\lambda(\nabla\cdot u)\,d\Omega - \int_{\partial\Omega} w\cdot\lambda(\nabla\cdot u)\cdot\underline{n}\,d\Gamma =$$

$$= \int_\Omega w\cdot\rho b\,d\Omega$$

* Recalling that $\varepsilon = \nabla^s u = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$, so ∇u and $\nabla\cdot u$ have to be square integrable.

$$u \in H^1(\Omega) \quad ; \quad u|_{\partial\Omega} = 0$$

$$w \in H^1(\Omega) \quad ; \quad w|_{\partial\Omega} = 0$$

* 2^{nd} equation.

$$\int_\Omega w\,(-\mu\Delta u-(\lambda+\mu)\nabla(\nabla\cdot u))\,d\Omega=\int w\cdot\rho b\,d\Omega$$

* Integrating by parts

$$\int_\Omega \nabla w\,\mu\nabla u\,d\Omega-\int_{\partial\Omega} w\cdot\mu\nabla u\,n\,d\Gamma+\int_\Omega \nabla w\cdot(\lambda+\mu)$$

$$(\nabla\cdot u)\,d\Omega-\int_{\partial\Omega} w\,(\lambda+\mu)(\nabla\cdot u)\cdot n\,d\Gamma=\int_\Omega w\cdot\rho b\,d\Omega$$

* Due to $\int \nabla w\,\mu\nabla u\,d\Omega$

* u and $w\in H^1(\Omega)$

* 3^{rd} equation

$$\int_\Omega w\,(\mu\nabla\times(\nabla\times u)-(\lambda+2\mu)\nabla(\nabla\cdot u))\,d\Omega=\int w\,\rho b\,d\Omega$$

* Integrating by parts and recovering the same identity than for the Maxwell problem:

$$\int_\Omega \nabla\times w\cdot\mu(\nabla\times u)\,d\Omega-\int_{\partial\Omega} w\,\mu(\nabla\times u)\cdot n\,d\Gamma+$$

$$+\int_\Omega \nabla w\,(\lambda+2\mu)(\nabla\cdot u)\,d\Omega-\int_{\partial\Omega} w\cdot(\lambda+2\mu)(\nabla\cdot u)\cdot n\,d\Gamma=$$

$$=\int_\Omega w\,\rho b\,d\Omega$$

* As $\nabla \times u$ and $\nabla \cdot u$ have to be square integrable:

$w \in H^1(\Omega)$; $w|_{\partial\Omega} = 0$

$u \in H^1(curl, \Omega) \cap H^1(div, \Omega)$; $u|_{\partial\Omega} = 0$

$u \in H^1(curl, H)$ implies $u_{i,j} \in H^1(\Omega)$ for $i \neq j$

$u \in H^1(div, H)$ implies $u_{i,j} \in H^1(\Omega$ for $i = j$

b)

It is written the weak form for the two subdomains splitted by the interface Γ, they are summed and compared with the weak form of the full domain, the extra terms should be 0.

* 1st equation

· Domain weak form splitted

$$\int_{\Omega_1} w \left(-2\mu \nabla \cdot (\varepsilon(u)) - \lambda \nabla(\nabla \cdot u) \right) d\Omega + \int_{\Omega_2} w \cdot (-2\mu \nabla \cdot \varepsilon(u)) -$$

$$- \lambda \nabla(\nabla \cdot u) \, d\Omega = \int_{\Omega} u \cdot \rho b \, d\Omega$$

Integrating by parts LHS:

$$= \int_{\Omega_1} \nabla w : 2\mu\, (\varepsilon(u))\, d\Omega - \int_{\Gamma} w\ 2\mu\, (\varepsilon(u))|_{\Omega_1} \cdot n\, d\Gamma +$$

$$+ \int_{\Omega_1} \nabla w \cdot \lambda\, (\nabla \cdot u)\, d\Omega - \int_{\Gamma} w \cdot \lambda\, (\nabla \cdot u|_{\Omega_1}) \cdot n\, d\Gamma +$$

$$+ \int_{\Omega_2} \nabla w : 2\mu\, (\varepsilon(u))\, d\Omega - \int_{\Gamma} w\ 2\mu\, \varepsilon(u)|_{\Omega}\, n\, d\Gamma +$$

$$+ \int_{\Omega_2} \nabla w\ \lambda(\nabla \cdot u)\, d\Omega - \int_{\Gamma} w \cdot \lambda\, (\nabla \cdot u|_{\Omega}) \cdot n\, d\Gamma$$

Transmission conditions:

$$[[\mu\, (\varepsilon(u)) \cdot n]] = 0$$

$$[[\lambda\, (\nabla \cdot u)\, n]] = 0$$

2^{nd} equation:

$$\int_{\Omega_1} w\, (-\mu \Delta u - (\lambda + \mu)\, \nabla\, (\nabla \cdot u)\, d\Omega + \int_{\Omega_2} w \cdot (-\mu \Delta u -$$

$$-(\lambda+\mu)\, \nabla\, (\nabla \cdot u)\,)\, d\Omega = \int w\, \rho\, b\ d\Omega$$

* Integrating by parts LHS

$$= \int_{\Omega_1} \nabla w \cdot \mu \nabla u \, d\Omega - \int_{\Gamma} w \cdot \mu \nabla u|_{\Omega_1} \cdot n \, d\Gamma +$$

$$+ \int_{\Omega_1} \nabla w \cdot (\lambda + \mu)(\nabla \cdot u) \, d\Omega - w(\lambda+\mu)(\nabla \cdot u|_{\Omega_1}) n \, d\Gamma$$

$$+ \int_{\Omega_2} \nabla w \cdot \mu \nabla u \, d\Omega - \int_{\Gamma} w \cdot \mu \nabla u|_{\Omega_2} \cdot n \, d\Gamma +$$

$$+ \int_{\Omega_2} \nabla w (\lambda+\mu)(\nabla \cdot u) \, d\Omega - \int_{\Gamma} w \cdot (\lambda+\mu)(\nabla \cdot u|_{\Omega_2}) n \, d\Gamma$$

* Transmission conditions

$$[[\mu \nabla u \cdot n]] = 0$$

$$[[(\lambda+\mu)(\nabla \cdot u) \cdot n]] = 0$$

- 3^{rd} equation

$$\int_{\Omega_1} w \cdot (\mu \nabla \times (\nabla \times u) - (\lambda + 2\mu) \nabla (\nabla \cdot u)) \, d\Omega +$$

$$+ \int_{\Omega_2} w \cdot (\mu \nabla \times (\nabla \times u) - (\lambda + 2\mu) \nabla (\nabla \cdot u)) \, d\Omega =$$

$$= \int_{\Omega} w \rho f \, d\Omega$$

Integrating by parts LHS:

$$= \int_{\Omega_1} \nabla \times w \cdot \mu (\nabla \times u)\, d\Omega - \int_{\Omega_2} w \cdot \mu (\nabla \times u|_{\Omega_1}) \cdot \underline{n}\, d\Gamma +$$

$$+ \int_{\Omega_1} \nabla w \cdot (\lambda + 2\mu)(\nabla \cdot u)\, d\Omega - \int_{d\Omega} w \cdot (\lambda + 2\mu)(\nabla \cdot u|_{\Omega_1}) \cdot \underline{n}\, d\Gamma$$

$$+ \int_{\Omega_2} \nabla \times w \cdot \mu (\nabla \times u)\, d\Omega - \int_{d\Omega} w \cdot \mu (\nabla \times u|_{\Omega_2}) \cdot \underline{n}\, d\Gamma +$$

$$+ \int_{\Omega_2} \nabla w \cdot (\lambda + 2\mu)(\nabla \cdot u)\, d\Omega - \int_{d\Omega} w \cdot (\lambda + 2\mu)$$

$$(\nabla \cdot u|_{\Omega_2}) \cdot n\, d\Gamma$$

Transmission conditions are:

$$[[\mu (\nabla \times u) \cdot \underline{n}]] = 0$$

$$[[(\lambda + 2\mu)(\nabla \cdot u) \cdot \underline{n}]] = 0$$

2. Domain decomposition methods

1. Problem 1 section 1. Let $[0,L] = [0, L_1] \cup [L_2, L)$ with $L_2 < 1$

a) The iteration-by-subdomain scheme based on a Schwarz additive Domain Decomposition Method.

+ 1st subdomain

$$EI \frac{d^4 v_1^k}{dx^4} = f \text{ in } [0, L_1]$$

$$v_1^k(0) = \bar{v}_0 = 0$$

$$\frac{d v_1^k}{dx}(0) = \bar{v}_{x0} = 0$$

$$v_1^k(L_1) = v_2^{(k-1)}(L_1)$$

$$\frac{d v_1^k}{dx}(L_1) = \frac{d v_2^{(k-1)}}{dx}(L_1)$$

+ 2nd subdomain

$$EI \frac{d^4 v_2^k}{dx^4} = f \text{ in } [L_2, L]$$

$$v_2^k(L) = \bar{v}_L = 0$$

$$\frac{d v_2^k}{dx}(L) = \bar{v}_{xL} = 0$$

$$v_2^k(L_2) = v_1^l(L_2)$$

$$\frac{d v_2^k}{dx}(L_2) = \frac{d v_1^l}{dx}(L_2$$

Where,

$$\begin{cases} l = k-1 & \text{for Jacobi scheme} \\ l = k & \text{for Gauss-Seidel} \end{cases}$$

b)

Discretizing:

$$v \simeq v_h = \sum_{i=0}^{n} N v$$

$$\theta = \frac{dv}{dx} \simeq \theta_h$$

$$S_v = \text{span}\{N_i\}$$

\# The problem

$$EI \int_0^L \frac{d^2 \delta u_h}{dx} \frac{d^2 u_h}{dx^2} = \int_0^L \delta u\, f$$

$$\# \quad EI \int_0^L \frac{d^2 N^T}{dx} \frac{d^2 N}{dx^2} u_h = \int_0^L N^T f$$

$$\# \quad \boxed{A\, u = f}$$

\# Once space has been discretized:

$$\begin{bmatrix} A_{11_{11}} & A_{1L_{11}} \\ A_{L1} & A_{LL_{(1)}} \end{bmatrix} \begin{bmatrix} v_{1(1)} \\ v_{L(1)} \end{bmatrix} = \begin{bmatrix} f_{I(1)} \\ f_{L1} \end{bmatrix}$$

Step 1.a

Solving for $v_{I(1)}$

$$A_{11}^{(1)} v_I^{(1)(k)} + A_{IL_1}^{(1)} v_{L_1}^{(k)} = f_{I_1}$$

$$v_{I_{(1)}}^{k} = \left(A_{11}^{(1)}\right)^{-1} \left(f_I^{(1)} - A_{IL_1} v_{L_1}^{(k)} \right)$$

$$\left[\text{Where } v_{L_1}^{(k)} = v_{I2}^{(k-1)} \Big|_{L_1} \right]$$

+ Same thing to solve first derivatives (rotations):

$$\begin{bmatrix} \tilde{A}_{II_1} & \tilde{A}_{IL_1} \\ \tilde{A}_{L_1I_1} & \tilde{A}_{L_1L_1} \end{bmatrix} \begin{bmatrix} v'_{I_1} \\ v'_{L_1} \end{bmatrix} = \begin{bmatrix} \tilde{f}_I^{(1)} \\ \tilde{f}_{L_1} \end{bmatrix}$$

Step 1.b

$$\tilde{A}_{I_1I_1} v'^{(k)}_{I_1} + \tilde{A}_{I_1L_1} v'^{(k)}_{L_1} = \tilde{f}_{I_1}$$

$$v'^{(k)}_{I_{(1)}} = \left(\tilde{A}_{I_1I_1}\right)^{-1} \left(\tilde{f}_{I_1} - \tilde{A}_{I_1L_1} v'^{(k)}_{L_1} \right)$$

$$\text{where } v'^{(k)}_{L_1} = v_{I_2}^{(k-1)} \Big|_{L_2}$$

Step 1.a and 1.b are inside first subdomain.

* For the subdomain Ω_2, same thing can be done in a parallel form using Jacobi method.

Step 2a

$$v_{I_2}^{(k)} = A_{I_2 I_2}^{-1} \left(f_{I_2} - A_{I_2 L_2} v_{L_2}^{(k)} \right)$$

Where $v_{L_2}^{(k)} = v_{I_1}^{(k-1)} \Big|_{L_2}$

Step 2b

$$v'^{(k)}_{I_2} = \tilde{A}_{I_2 I_2}^{-1} \left(\tilde{f}_{I_2} - \tilde{A}_{I_2 L_2} v'^{(k)}_{L_2} \right)$$

Where $v'^{(k)}_{L_2} = v_{I_1}^{(k-1)} \Big|_{L_2}$

2. Consider Problem 2 of Section 1

a) Dirichlet - to - Neumann coupling

The domain is divided in two non-intersecting subdomains

$$\Omega_1 \cup \Omega_2 = \Omega$$

$$\Omega_1 \cap \Omega_2 = \Gamma_{12}$$

$$\Gamma_i = \partial \Omega \cap \Omega_i$$

Subdomain is solved as a Neumann problem, enforcing the transmission condition of fluxes

$$\nu \nabla \times \nabla \times u_1^k = f \text{ in } \Omega_1$$
$$n \times u_1^k = 0 \quad \text{on } \Gamma_1$$
$$\nu_1 (\nabla \times u_1^k) \cdot n = \nu_2 (\nabla \times u_2^{(k-1)}) \cdot n \text{ on } \Gamma_{12}$$

Subdomain 2 is solved as a Dirichlet problem enforcing the transmission of the variable.

$$\nu \nabla \times \nabla \times u_2^k = f \quad \text{in } \Omega_2$$
$$n \times u_2^k = 0 \quad \text{on } \Gamma_2$$
$$n \times u_2^k = n \times u_1^l \quad \text{on } \Gamma_{12}$$

Where $l = \begin{cases} k-1 & \text{for Jacobi} \\ k & \text{for Gauss-Seidel} \end{cases}$

b) Steklov - Poincaré operator

The problem can be expressed as the sum of a problem where the solution is 0 at the boundary and a homogenous problem with the correct boundary conditions.
The solution is splitted at each subdomain

$$u_i = u_i^0 + \tilde{u}_l \quad \text{for } i = 1,2$$

+

$$\nu \nabla \times \nabla \times u_i^0 = f \text{ in } \Omega_i$$
$$n \times u_i^0 = 0 \text{ on } \Gamma_i$$
$$n \times u_i^0 = 0 \text{ on } \Gamma_{12}$$

$$\nu \nabla \times \nabla \times \tilde{u}_l = 0 \text{ in } \Omega_i$$
$$n \times \tilde{u}_l = 0 \text{ on } \Gamma_i$$
$$n \times \tilde{u}_l = \varphi \text{ on } \Gamma_{12}$$

\# With the transmission of the variable:

$$[[n \times u]] = 0$$

\# For the fluxes

$$[[\nu (\nabla \times u) \cdot n]] = 0$$

\+ Find u on Γ_{12} to assure transmission conditions

$$\nu_1 (\nabla \times u_1) \cdot n = \nu_2 (\nabla \times u_2) \cdot n \Rightarrow$$

$$\Rightarrow \nu_1 (\nabla \times \tilde{u}_1) \cdot n - \nu_2 (\nabla \times \tilde{u}_2) \cdot n =$$

$$= -\nu_1 (\nabla \times u_1^0) \cdot n + \nu_2 (\nabla \times u_2^0) n$$

\# Steklov - Poincaré operator is

$$S : H^{1/2}(\mathrm{curl}, \Gamma_{12}) \to H^{-1/2}(\mathrm{curl}, \Gamma_{12})$$

$$\varphi \to \nu_1 (\nabla \times \tilde{u}_1) \cdot n - \nu_2 (\nabla \times \tilde{u}_2) \cdot n$$

$$G = -\nu_1 (\nabla \times u_1^0) \cdot n + \nu_2 (\nabla \times u_2^0) \cdot n$$

\# To ensure transmission conditions:

$$\boxed{S \varphi = G}$$

c)

* Neumann problem for 1st subdomain

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} U_1 \\ U_\Gamma \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2} U_2 \end{bmatrix}$$

* Dirichlet problem for 2^{nd} subdomain

$$A_{22} U_2 = F_2 - A_{2\Gamma} U_\Gamma$$

\# Therefore

$$\begin{bmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_\Gamma \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma \\ F_2 \end{bmatrix}$$

* Iterative scheme for Neumann problem

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{(k)} \\ u_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2} U_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)} U_\Gamma^{(k-1)} \end{bmatrix}$$

* Iterative scheme for Dirichlet problem:

$$A_{22} U_2^k = F_2 - A_{2\Gamma} U_\Gamma^l$$

where l depends on if it is Jacobi or Gauss-Seidel scheme

3.

$$-k \Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

a) Dirichlet - Robin $(\Omega_1, \Omega_2, \Gamma_1, \Gamma_2, \Gamma_{12})$

* Dirichlet problem

$$-k \Delta u_1^k = f \quad \text{in } \Omega_1$$
$$u_1^k = 0 \quad \text{on } \Gamma_1$$
$$u_1^k = u_2^{(k-1)} \quad \text{on } \Gamma_{12}$$

* Robin problem:

$$-k\, \Delta u_2^k = f \quad in\ \Omega_2$$
$$u_2^k = 0 \qquad on\ \Gamma_2$$

$$k_2 (\nabla u_2^k \cdot n) + \gamma\, u_2^k = k_2 (\nabla u_1^l \cdot n) + \gamma\, u_1^l \quad on\ \Gamma_{12}$$

Where $\gamma > 0$ and $l = k-1$ for Jacobi and $l = k$ for Gauss-Seidel scheme

b)

Once is discretized in space

\# Dirichlet problem:

$$A_{22}\, U_2 = F_2 - A_{2\Gamma}\, U_\Gamma$$

\# Robin problem

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} + \gamma I \end{bmatrix} \begin{bmatrix} U_1 \\ U_\Gamma \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2}\, U_2 - (A_\Gamma^{(2)} - \gamma I)\, U_\Gamma \end{bmatrix}$$

Matrix $A_{\Gamma 1}$ includes terms of Robin BC

\# Iterative scheme

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} + \gamma I \end{bmatrix} \begin{bmatrix} U_1^k \\ U_\Gamma^k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 1}\, U_2^{(k-1)} - (A_{\Gamma\Gamma}^{(2)} - \gamma I)\, U_\Gamma^{(k-1)} \end{bmatrix}$$

$$A_{22}\, U_2^k = F_2 - A_{2\Gamma}\, U_\Gamma^l$$

c) Schur complement

+ local problems:

$$-K \Delta u_i^0 = f \quad \text{in } \Omega_i$$
$$u_i^0 = 0 \quad \text{on } \Gamma_i$$
$$u_i^0 = 0 \quad \text{on } \Gamma_{12}$$

+

$$G = -k_1 (\nabla u_1^0 \cdot n) + k_2 (\nabla u_2^0 \cdot n)$$

+ Discrete version.

$$A_{11} U_1^0 = F_1$$
$$A_{22} U_2^0 = F_2$$

+ Steklov - Poincaré operator, given the trace of u on Γ_{12} (φ)

$$-K \Delta \tilde{u}_i = 0 \quad \text{in } \Omega_i$$
$$\tilde{u}_i = 0 \quad \text{on } \Gamma_i$$
$$\tilde{u}_i = \varphi \quad \text{on } \Gamma_{12}$$

+ Discretizing

$$A_{11} \tilde{U}_1 + A_{1\Gamma} U_\Gamma = 0$$
$$A_{22} \tilde{U}_2 + A_{2\Gamma} U_\Gamma = 0$$

Where φ is discretized as φ

* $S\varphi = k_1 (\nabla \hat{u}_1 \cdot n) - k_2 (\nabla \hat{u}_2 \cdot n)$

$G = F_\Gamma - A_{\Gamma 1} U_1^0 - A_{\Gamma 2} U_2^0$

* Substituting U_1^0 and U_2^0:

$G = F_\Gamma - A_{\Gamma 1} A_{11}^{-1} F_1 - A_{\Gamma 2} A_{22}^{-1} F_2$

$S\varphi = A_{\Gamma\Gamma} U_\Gamma + A_{\Gamma 1} \hat{U}_1 + A_{\Gamma 2} \hat{U}_2$

$S\varphi = (A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}) U_\Gamma$

* Continuous problem:

$$S\varphi = G$$

* The final discrete form:

$$(A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}) U_\Gamma = F_\Gamma - A_{\Gamma 1} U_1^0 - A_{\Gamma 2} U_2^0$$

d) Schur complement

$S = S_1 + S_2$

$S_1 = A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}$

$S_2 = A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}$

+ Considering Gauss-Seidel scheme

$$U_1^k = A_{11}^{-1} (F_1 - A_{1\Gamma} U_\Gamma^k)$$

$$U_2^k = A_{22}^{-1} (F_2 - A_{2\Gamma} U_\Gamma^k)$$

$$A_{\Gamma 1} A_{11}^{-1} (F_1 - A_{1\Gamma} U_\Gamma^k) + A_{\Gamma\Gamma}^{(1)} U_\Gamma^k + \gamma U_\Gamma^k =$$

$$= F_\Gamma - A_{\Gamma 1} A_{22}^{-1} (F_2 - A_{2\Gamma} U_\Gamma^{(k-1)}) - (A_{\Gamma\Gamma}^{(2)} - \gamma I) U_\Gamma^{(k-1)}$$

$$(S_1 + \gamma I) U_\Gamma^k = (F_\Gamma - A_{\Gamma 1} A_{11}^{-1} F_1 - A_{\Gamma 1} A_{22}^{-1} F_2) -$$

$$- (A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 1} A_{22}^{-1} A_{2\Gamma} + A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}) U_\Gamma^{(k-1)} +$$

$$+ (A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}) U_\Gamma^{(k-1)} + \gamma (A_{22}^{-1} U_\Gamma^{(k-1)})$$

$$(S_1 + \gamma I) U_\Gamma^k = G - S U_\Gamma^{(k-1)} + (S_1 + \gamma I) U_\Gamma^{(k-1)}$$

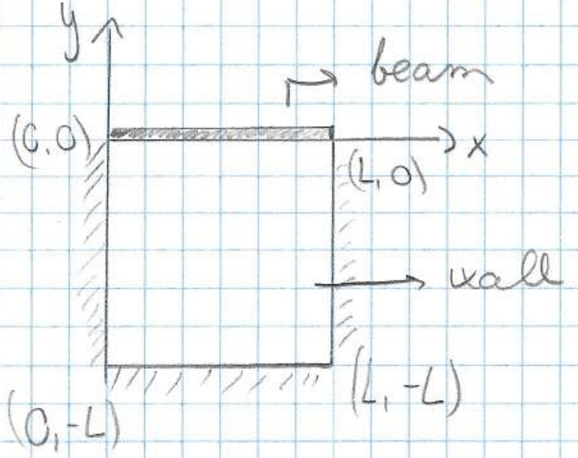
$$\boxed{U_\Gamma^k = U_\Gamma^{(k-1)} + (S_1 + \gamma I)^{-1} (G - S U_\Gamma^{(k-1)})}$$

+ The preconditioner of Dirichlet-Robinson:

$$\boxed{P_{DR} = (S_1 + \gamma I)^{-1}}$$

3 Coupling of heterogenous problems

1. Beam described in Problem 1, section 1



a) * It is supposed plane stress hypothesis

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$$

* As there are not body forces applied on the wall, the equation of equilibrium: $(b = 0) \Rightarrow \underline{\nabla} \cdot \underline{\underline{\sigma}} + b = 0$

$$\nabla \cdot \sigma = 0 \quad \text{in } \Omega_{wall} \quad \text{where } \sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

* Constitutive law:

$$\underline{\sigma} = \underline{C}\,\underline{\varepsilon}$$

Where

$$C = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \quad ; \quad \sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

↳ Constitutive matrix

Strain:

$$\varepsilon = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) \Rightarrow \varepsilon = \begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{bmatrix} = \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\end{bmatrix}$$

Therefore, the behaviour of the wall displacement is:

$$\nabla \cdot \left(\frac{C}{2} \left(\nabla u + (\nabla u)^T\right)\right) = 0 \quad \text{in } \Omega \text{ wall}$$

The BCs are the null displacements on the clamped sides, denoted as Γ_D (lateral and bottom sides):

$$u = 0 \quad \text{on } \Gamma_D = 0$$

At the upper side, the transmission conditions have to be enforced (traction forces of the beam).

(2) b)

Now, it is considered the contribution of the force term from the wall:

$$EI \frac{d^4 v}{dx^4} = f - t^y_{wall}\Big|_{y=0} \quad \text{in } \Omega \text{ beam}$$

The term is negative because t wall is opposite to the distributed load of the beam (f)

* The boundary conditions of beam are

$$v(0,0) = \frac{dv}{dx}(0,0) = v(L,0) = \frac{dv}{dx}(L,0) = 0$$

$$u_{beam}(0,0) = u_{wall}(0,0)$$

c) Transmission conditions for v at the interface (Γ) between the wall and the beam situated on $y=0$.
The vertical displacement (v) must be the same in the wall and in the beam.
Therefore, it is computed the jump of v across the interface, enforcing the continuity of v)

1. $$[[v]]_\Gamma = 0 \Rightarrow v_{wall}(x,0) = v_{beam}(x)$$

* The normal component of the traction from the beam is:

$$t^x_{beam} = EA \frac{du_{wall}}{dx}$$

* It is assumed that $\tau_{xy} = 0$, so the normal traction is in the vertical direction

$$t^y_{wall}(x) = [0,1] \cdot (\underline{\sigma}(x,0) \cdot \underline{n})$$

$$t^y_{wall} = (\sigma_{yy})_{wall}$$

* The second transmission is to enforce continuity of tractions.

2. $[[t^y \cdot \underline{n}]] = 0 \Rightarrow t^y_{wall} \cdot \underline{n} = - t^y_{beam} \cdot \underline{n}$

d) As it shown for v, in horizontal displacement (u) the continuity must be enforced

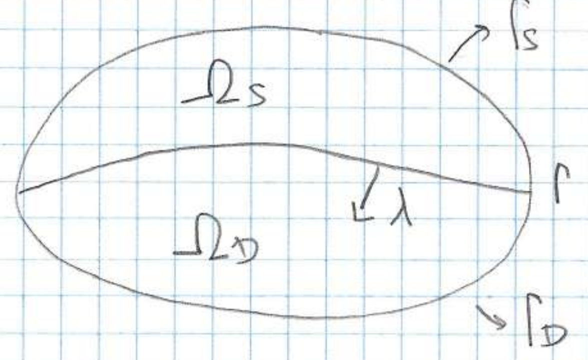
$$u_{beam}(x) = u_{wall}(x, 0) = 0$$

The tangential component of the traction on the wall at $y = 0$ should:

$$(\tau_{xy})_{wall} = 0$$

Euler-Bernoulli beam theory would not be valid if the horizontal displacement (u) and the traction (τ_{xy}) were not assumed to be 0.

2.



S: Stokes problem

D: Darcy problem

$\lambda = n \cdot u_S$

a)

Stokes problem:

$$-\nu \Delta u_s + \nabla p_s = f \quad \text{in } \Omega_s$$
$$\nabla \cdot u_s = 0 \quad \text{in } \Omega_s$$
$$u_s = \bar{u}_s \quad \text{on } \Gamma_s$$

Darcy problem

$$K^{-1} u_D + \nabla \varphi = 0 \quad \text{in } \Omega_D$$
$$\nabla \cdot u_D = 0 \quad \text{in } \Omega_D$$
$$u_D = \bar{u}_D \quad \text{on } \Gamma_D$$

Transmission conditions on $\Gamma \Rightarrow S_S(\lambda) = S_D(\lambda)$

$\underline{n} \cdot u_s = n \cdot u_D \rightarrow$ "Strongly"

$\left.\begin{array}{l} p_s I - \nu\, n \cdot \nabla u_s \cdot n = \varphi \\ u_s \cdot t + \dfrac{\sqrt{K}}{\alpha_{BJ}} \nu t \cdot (n \cdot \nabla u_s) = 0 \end{array}\right\} \rightarrow$ "Weakly"

Where $\underline{t}$ is a unit tangential vector on Γ

Discretization of Stokes problem applying weighting function (w_s for momentum and q_s for the incompressibility)

$$\int_\Omega w_s \cdot (-\nu \Delta u_s + \nabla p)\, d\Omega = \int_\Omega w_s \cdot f\, d\Omega$$

$$\int q_s \cdot (\nabla \cdot u_s)\, d\Omega = 0$$

Integrating by parts:

$$\int_{\Omega_s} w_s\left(-\nu\Delta u_s + \nabla p_s\right) = \int_{\Omega_s} \nu \nabla w_s : \nabla u_s - \int_{\Omega} p_s \nabla\cdot w_s$$

$$- \int_{\partial\Omega_s} w_s \left[n\cdot\left(-p_s I + \nu\nabla u_s\right)\right]$$

Yielding to:

$$\int_{\Omega_s} \nabla w_s : \nu \nabla u_s \, d\Omega - \int_{\Omega_s} (\nabla\cdot w_s)\, p_s \, d\Omega = \int_{\Omega_s} w_s \, f \, d\Omega +$$

$$+ \int_{\partial\Omega_s} w_s \left[(\nu\nabla u_s - pI)\cdot n\right] d\Gamma$$

The linear system is:

$$\begin{bmatrix} K & G_s \\ G_s^T & 0 \end{bmatrix}\begin{bmatrix} u_s \\ p \end{bmatrix} = \begin{bmatrix} f_s \\ 0 \end{bmatrix}$$

Having substituted by:

$$u_s^h = \sum N_i u_i \qquad w_s = \text{span}\{N_j\}$$

$$p_s^h = \sum \hat{N}_i p_i \qquad q_s = \text{span}\{\hat{N}_j\}$$

The space functions are

$$u_s \in H^1(\Omega_s) \qquad p \in L_2(\Omega_s)$$

* Discretization of Darcy problem, multiplying by weighting functions w_D and q_D:

$$\int_{\Omega_D} w_D \cdot (K^{-1} u_D + \nabla \varphi) \, d\Omega = 0$$

$$\int_{\Omega_D} q_D \cdot (\nabla \cdot u_D) \, d\Omega = 0$$

* Integrating by parts:

$$\int_{\Omega_D} w_D (K^{-1} u_D + \nabla \varphi) \, d\Omega = \int_{\Omega_D} w_D K^{-1} u_D \, d\Omega -$$

$$- \int_{\Omega_D} (\nabla \cdot w_D) \cdot \varphi \, d\Omega + \int_{\partial\Omega_D} w_D \cdot \varphi \cdot n \, d\Gamma$$

* linear system:

$$\begin{bmatrix} M & G_D \\ G_D^T & 0 \end{bmatrix} \begin{bmatrix} u_S \\ \varphi \end{bmatrix} = \begin{bmatrix} f_D \\ 0 \end{bmatrix}$$

*

$u_D^h = \sum N_i u_i$ $\qquad$ $w_D \equiv \text{span}\{N_j\}$

$\varphi = \sum \hat{N}_i \varphi_i$ $\qquad$ $q_D \equiv \text{span}\{\hat{N}_j\}$

* The Steklov - Poincaré problem is written as:

$$S_S(\lambda) = S_D(\lambda)$$

* Its discretization yeld to the linear system

$$\begin{bmatrix} A_{SS} & A_{SD} \\ A_{DS} & A_{DD} \end{bmatrix} \begin{bmatrix} u_S^{\Gamma} \\ u_D^{\Gamma} \end{bmatrix} = \begin{bmatrix} f_S \\ f_D \end{bmatrix}$$

b) Dirichet - Neumann iteration:

* The Stokes problem is solved using Neumann BC, with a given guess of the velocity at the interface

$$\begin{bmatrix} K_{SS} & K_{S\Gamma} & G_S \\ K_{\Gamma S} & K_{\Gamma\Gamma} & G_{\Gamma S} \\ G_S^T & G_{\Gamma p}^T & 0 \end{bmatrix} \begin{bmatrix} u_S^k \\ u_\Gamma^k \\ p^k \end{bmatrix} = \begin{bmatrix} f_S \\ f_\Gamma - M_{\Gamma D} u_D^{(k-1)} - G_{\Gamma\varphi} \varphi^{(k-1)} - M_{\Gamma\Gamma} u_\Gamma^{(k-1)} \\ 0 \end{bmatrix}$$

* The Darcy problem is solved using Dirichet BC.

$$\begin{bmatrix} M_{DD} & G_D \\ G_D^T & 0 \end{bmatrix} \begin{bmatrix} u_D^k \\ \varphi^k \end{bmatrix} = \begin{bmatrix} f_D - A_{D\Gamma} u_\Gamma^k \\ 0 \end{bmatrix}$$

* It is used a Gauss - Seidel scheme

c)

Richardson iteration:

From the Neumann problem, p and u_s^k are in function of u_Γ^k

$$\begin{bmatrix} K_{ss} & G_s \\ G_s^T & 0 \end{bmatrix}\begin{bmatrix} u_s^k \\ p^k \end{bmatrix} = \begin{bmatrix} f_s - K_{s\Gamma} u_\Gamma^k \\ -G_{\Gamma p}^T u_\Gamma^k \end{bmatrix} = \begin{bmatrix} f_s \\ 0 \end{bmatrix} - \begin{bmatrix} K_{s\Gamma} \\ G_{\Gamma p}^T \end{bmatrix} u_\Gamma^k$$

From Dirichet problem

$$\begin{bmatrix} M_{DD} & G_D \\ G_D^T & 0 \end{bmatrix}\begin{bmatrix} u_D^k \\ \varphi^k \end{bmatrix} = \begin{bmatrix} f_D - A_{D\Gamma} u^k \\ 0 \end{bmatrix} = \begin{bmatrix} f_D \\ 0 \end{bmatrix} - \begin{bmatrix} A_{D\Gamma} \\ 0 \end{bmatrix} u_\Gamma^k$$

From Neumann equation of b)

The second row:

$$K_{\Gamma s} u_s^k + K_{\Gamma\Gamma} u_\Gamma^k + G_{\Gamma p} p^k = f_\Gamma - M_{\Gamma D} u_D^{(k-1)} - G_{\Gamma\varphi} \varphi^{(k-1)} - M_{\Gamma\Gamma} u_\Gamma^{(k-1)};$$

$$\begin{bmatrix} K_{\Gamma s} & G_{\Gamma p} \end{bmatrix}\begin{bmatrix} u_s^k \\ p^k \end{bmatrix} + K_{\Gamma\Gamma} u_\Gamma^k = f_\Gamma - \begin{bmatrix} M_{\Gamma D} & G_{\Gamma\varphi} \end{bmatrix}\begin{bmatrix} u_D^{(k-1)} \\ \varphi^{(k-1)} \end{bmatrix} - M_{\Gamma\Gamma} u_\Gamma^{(k-1)}$$

$$[K_{rs} \quad G_{rp}] \begin{bmatrix} K_{ss} & G_s \\ G_s^T & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} f_s \\ 0 \end{bmatrix} - \begin{bmatrix} K_{sr} \\ G_{rp}^T \end{bmatrix} u_r^k \right) + K_{rr} u_r^k =$$

$$= f_r - [M_{rD} \quad G_{rq}] \begin{bmatrix} M_{DD} & G_D \\ G_D^T & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} f_D \\ 0 \end{bmatrix} - \begin{bmatrix} A_{D r} \\ 0 \end{bmatrix} u^{(k-1)} \right)$$

* Finally, isolating u_r^k:

$$\left(K_{rr} - [K_{rs} \quad G_{rp}] \begin{bmatrix} K_{ss} & G_s \\ G_s^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} K_{sr} \\ G_{rp}^T \end{bmatrix} \right) u_r^k =$$

$$= f_r - [K_{rs} \quad G_{rp}] \begin{bmatrix} K_{ss} & G_s \\ G_s^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} f_s \\ 0 \end{bmatrix} -$$

$$- [M_{rD} \quad G_{rq}] \begin{bmatrix} M_{DD} & G_D \\ G_D^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} f_D \\ 0 \end{bmatrix} +$$

$$+ [M_{rD} \quad G_{rq}] \begin{bmatrix} M_{DD} & G_D \\ G_D^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{Dr} \\ 0 \end{bmatrix} u_r^{(k-1)}$$

* Which has the same structure of Richardson method:

$$\boxed{U_r^k = U^{(k-1)} + B^{-1} R^k}$$

4. Monolithic and partitioned schemes in time

One - dimensional, transient, heat transfer equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - k \dfrac{\partial^2 u}{\partial x^2} = f \quad \text{in } [0,1] \\ u(x=0, t) = 0 \\ u(x=1, t) = 0 \\ u(x, t=0) = 0 \end{cases}$$

1. BDF1 scheme for the discretization in time.
Considering $k = 1$, $f = 1$, $\delta t = 1$

* Discretization of the variable u and $\frac{\partial u}{\partial t}$

$$u^h(x,t) = \sum_{i=0}^{n} u_i(t) N_i(x)$$

$$\partial_t u^h(x,t) = \sum_{i=0}^{n} \partial_t u_i(t) N_i(x)$$

* The weak form, with the weighting function w, is:

$$\int_0^1 w^h \cdot (\partial t\, u^h - k \Delta u^h) \, dx = \int_0^1 w^h \cdot f \, dx$$

* Integrating by parts, it is obtained a 1st order equation:

$$\int_0^1 w^h \cdot \partial_t u^h \, dx + \int_0^1 \nabla w^h \cdot k \nabla u^h \, dx - \left[w^h k \nabla u^h \cdot \underline{n} \right]_0^1 =$$

$$= \int_0^1 w^h f \, dx$$

* As there is not Neumann BC:

$$\left[w^h k \nabla u^h \cdot \underline{n} \right]_0^1 = 0$$

* Considering $w^h = \text{span}\{N_j\}$ and substituting the numerical values, it is obtained:

$$\int_0^1 N_j \cdot \sum_{i=0}^{n} \partial_t u_i N_i \, dx + K \int_0^1 \nabla N_j \sum_{i=0}^{n} u_i(t) \nabla N_i(x) \, dx =$$

$$= \int_0^1 N_j \, dx \quad \text{for} \quad j = 1, \dots n-1$$

* The first and last node have not been considered in the equation due to the fact these values are prescribed by the null boundary integral (all BC are Dirichlet boundaries equal to 0.)

* Resulting the linear system:

$$\underline{M}\, \partial t\, u^h + \underline{K}\, u^h = f$$

Where,

$$M_{ij} = \int_0^1 N_i \cdot N_j \, dx \quad \Rightarrow \text{Mass matrix}$$

$$K_{ij} = \int_0^1 \nabla N_i \, \nabla N_j \, dx \quad \Rightarrow \text{Stiffness matrix}$$

$$f_i = \int_0^1 N_i \, dx \quad \Rightarrow \text{Source term}$$

* Discretization by Backward difference in time (BDF1) which is implicit and unconditionally stable $O(\delta t)$

$$\partial_t u_{n+1} = (u_{n+1} - u_n) / \delta t$$

* Yielding to, substituting:

$$\frac{M}{\delta t} (u_h^{n+1} - u_h^n) + K u_h^{n+1} = f$$

$$\left(\frac{M}{\delta t} + K\right) u_h^{n+1} = f - \frac{M}{\delta t} u_h^n$$

* Considering $\delta t = 1$

$$\left[(M + K)\, u_h^{n+1} = f + M u_h^n \right]$$

2. Monolithic scheme, split the domain into two sub-domains

$\Omega_1 = [0, 0.4]$ $\qquad$ $\Omega_2 = [0.4, 1]$

Γ at $x = 0.4$

* Sub-domain 1:

$$\int_0^{0.4} w_1^h \cdot \partial_t u_1^h \, dx + \int_0^{0.4} \nabla w_1^h \nabla u_1^h \, dx - \left[w_1^h k \nabla u_1^h \underline{n} \right]_\Gamma = $$
$$= \int_0^{0.4} w_1^h f \, dx$$

* Sub-domain 2:

$$\int_{0.4}^{1} w_2^h \, \partial_t u_2^h \, dx + \int_{0.4}^{1} \nabla w_2^h \nabla u_2^h \, dx - \left[w_2^h k \nabla u_2^h \cdot \underline{n} \right]_\Gamma =$$
$$= \int_{0.4}^{1} w_2^h f \, dx$$

* Transmission conditions:

$[[u]]_\Gamma = 0 \Rightarrow$ Strong continuity (continuity of temperature)

$\left[\left[k \frac{\partial u}{\partial x} n \right]\right] = 0 \Rightarrow$ Weak continuity (continuity of fluxes)

* In order to solve the problem in a monolithic way, the two equations for the two subdomains are summed:

$$\int_{0.4}^{1} w_2^h \cdot \partial_t u_2^h \, dx + \int_{0.4}^{1} \nabla w_2^h \, \nabla u_2^h \, dx + \int_{0}^{0.4} w_1^h \cdot \partial_t u_1^h +$$

$$+ \int_{0}^{0.4} w_1^h \, \nabla u_1^h \, dx \underbrace{- \left[w_1^h k \nabla u_1^h \right]_\Gamma + \left[w_2 \, k \nabla u_2^h \right]_\Gamma}_{=0} =$$

$$= \int_{0}^{0.4} w_1^h \, dx + \int_{0.4}^{1} w_2^h \, dx$$

The boundary integrals at the interface Γ are equal to 0 due to the second transmition ($\underline{n}_1 = -\underline{n}_2$) to assure the continuity of fluxes

Therefore, it is shown that no boundary integrals are required at the interface. It is also neglected the integral boundary at the Dirichet BC

3. Dirichlet - to - Neumann operator for left subdomain, given u_i^n at time step n, and interface value u_2^{n+1}.

* Assuming notation:

$$\underline{u}_I^{(1)} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \quad \underbrace{\underline{u}_\Gamma = u_2}_{\text{Interface}}, \quad u_I^{(2)} = \begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

↳ interior nodes of subdomain 1 ↳ interior nodes of subdomain 2

* Global problem:

$$\left(\frac{M}{\delta t} + K \right) u^{n+1} = f + \frac{M}{\delta t} u^n$$

* For subdomain 1, since it is monolithic solver:

$$\underbrace{\left(\frac{M^{(1)}}{\delta t} + K^{(1)} \right)}_{A^{(1)}} u_{(1)}^{n+1} = f_{(1)} + \frac{M^{(1)}}{\delta t} u_{(1)}^n$$

* Using $\delta t = 1$, it is written:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_I^{(1)\,n+1} \\ u_\Gamma^{\,n+1} \end{bmatrix} = \begin{bmatrix} f_I^{(1)} \\ f_\Gamma^{(1)} \end{bmatrix} + \begin{bmatrix} M_{II}^{(1)} & M_{I\Gamma}^{(1)} \\ M_{\Gamma I}^{(1)} & M_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_I^{(1)\,n} \\ u_\Gamma^{\,n} \end{bmatrix} + \begin{bmatrix} 0 \\ \phi^{n+1} \end{bmatrix}$$

$u_\Gamma^{n+1} = u_2^{n+1} \Rightarrow$ Value given as a Dirichlet BC

* To work out $u_I^{(1)\,n+1}$, having the u_Γ^{n+1} as a Dirichlet BC:

$$A_{II}^{(1)} U_I^{(1)n+1} + A_{I\Gamma}^{(1)} U_\Gamma^{n+1} = f_I^{(1)} + M_{II}^{(1)} U_I^{(1)n} + M_{I\Gamma}^{(1)} U_\Gamma^{n}$$

$$\left[U_I^{(1)\,n+1} = \left(A_{II}^{(1)}\right)^{-1} \left(f_I^{(1)} + M_{II}^{(1)} U_I^{(1)n} + M_{I\Gamma}^{(1)} U_\Gamma^{n} - A_{I\Gamma}^{(1)} U_\Gamma^{n+1} \right) \right]$$

↳ interior nodes of subdomain 1.

4. Neumann - to - Dirichlet operator for right subdomain, given u_i^n and interface value for flux $\phi^{n+1} = K \partial_x u^{n+1}$ at interface

* The right subdomain is solved as a Neumann problem where from the continuity of fluxes:

$$\phi_2^{n+1} = -\phi_1^{n+1} = -\phi^{n+1}$$ (considering it comes from the left subdomain)

* For subdomain 2:, considering $\delta t = 1$

$$\left(M^{(2)} + K^{(2)}\right) U_{(2)}^{n+1} = f_{(2)} - \phi_{(2)}^{n+1} + M^{(2)} U_{(2)}^{n}$$

where

$$\phi_{(2)}^{n+1} = \begin{bmatrix} 0_I^{(2)} \\ \phi^{n+1} \end{bmatrix} \begin{matrix} \to \text{interior nodes} \\ \to \text{at interface} \end{matrix}$$

* It yields to:

$$\begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} u_I^{(2)\,n+1} \\ u_\Gamma^{\,n+1} \end{bmatrix} = \begin{bmatrix} f_I^{(2)} \\ f_\Gamma^{(2)} \end{bmatrix} + \begin{bmatrix} 0 \\ -\phi^{n+1} \end{bmatrix} + \begin{bmatrix} M_{II}^{(2)} & M_{I\Gamma}^{(2)} \\ M_{\Gamma I}^{(2)} & M_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} u_I^{(2)\,n} \\ u_\Gamma^{\,n} \end{bmatrix}$$

* from 1st subdomain:

$$\phi^{n+1} = -f_\Gamma^{(1)} + A_{\Gamma\Gamma}^{(1)} u_\Gamma^{\,n+1} + A_{\Gamma I}^{(1)} u_I^{(1)\,n+1} - M_{\Gamma I}^{(1)} u_I^{(1)\,n} - M_{\Gamma\Gamma}^{(1)} u_\Gamma^{\,n}$$

*

$$A_{\Gamma I}^{(2)} u_I^{(2)\,n+1} + A_{\Gamma\Gamma}^{(2)} u_\Gamma^{\,n+1} = f_\Gamma^{(2)} - \phi^{n+1} + M_{\Gamma I}^{(2)} u_I^{(2)\,n} + M_{\Gamma\Gamma}^{(2)} u_\Gamma^{\,n}$$

* Therefore:

$$\left[u_\Gamma^{\,n+1} = (A_{\Gamma\Gamma}^{(2)})^{-1} \left(-A_{\Gamma I}^{(2)} u_I^{(2)\,n+1} + f_\Gamma^{(2)} - \phi^{n+1} + M_{\Gamma I}^{(2)} u_I^{(2)\,n} + M_{\Gamma\Gamma}^{(2)} u_\Gamma^{\,n} \right) \right]$$

* Having computed $u_\Gamma^{\,n+1}$, it is calculated $u_I^{(2)\,n+1}$ from the previous system of equations:

$$A_{II}^{(2)} u_I^{(2)\,n+1} + A_{I\Gamma}^{(2)} u_\Gamma^{\,n+1} = f_I^{(2)} + M_{II}^{(2)} u_I^{(2)\,n} + M_{I\Gamma}^{(2)} u_\Gamma^{\,n}$$

$$u_I^{(2)\,n+1} = (A_{II}^{(2)})^{-1} \left(-A_{I\Gamma}^{(2)} u_\Gamma^{\,n+1} + f_I^{(2)} + M_{II}^{(2)} u_I^{(2)\,n} + M_{I\Gamma}^{(2)} u_\Gamma^{\,n} \right)$$

* Substituting $u_\Gamma^{\,n+1}$, it is obtained $u_I^{(2)\,n+1}$

$$A_{II}^{(2)} u_I^{(2)\,n+1} + A_{I\Gamma}^{(2)} (A_{\Gamma\Gamma}^{(2)})^{-1} \Big(-A_{\Gamma I}^{(2)} u_I^{(2)\,n+1} + f_\Gamma^{(2)} - \phi^{n+1} +$$

$$+ M_{\Gamma I}^{(2)} u_I^{(2)\,n} + M_{\Gamma\Gamma}^{(2)} u_\Gamma^{\,n} \Big) = f_I^{(2)} + M_{II}^{(2)} u_I^{(2)\,n} + M_{I\Gamma}^{(2)} u_\Gamma^{\,n}$$

[* Where the only unknown is $u_I^{(2)\,n+1}$, so the interior nodes for the subdomain 2 are calculated]

5. In staggered approach, instead of solving the monolithic as one global system, two local problems (the two subdomains) are solved in a parallel scheme for the same time step.
It is used the 1st order of approximation for the predicted values.

\# For 1st subdomain (Dirichlet BC at interface):

$$\begin{cases} \dfrac{\partial u_{(1)}^{i+1}}{\partial t} - k \dfrac{\partial^2 u_{(1)}^{i+1}}{\partial x^2} = f_{(1)} & \text{in } \Omega_1 \\ u_{(1)}^{i+1} = u^i|_\Gamma & \text{on } \Gamma_{12} \Rightarrow 1^{st} \text{ order of approximation} \\ u_{(1)} = 0 & \text{on } (x=0) \end{cases}$$

\# $$\left[U_I^{(1)\, i+1,\, n+1} = \left(A_{11}^{(1)}\right)^{-1} \left(f_I^{(1)} + M_{11}^{(1)} U_1^{(1)\, n} + M_{11}^{(1)} U_I^{(1)\, n} + - A_{I\Gamma}^{(1)} U_\Gamma^{n+1,\, i} \right) \right]$$

\# For 2nd subdomain (Neumann BC at the interface)

$$\begin{cases} \dfrac{\partial u_{(2)}^{i+1}}{\partial t} - k \dfrac{\partial^2 u_{(2)}^{i+1}}{\partial x^2} = f_{(2)} \\ k \dfrac{\partial u_{(2)}^{i+1}}{\partial x} = -k \dfrac{\partial u_{(1)}^{i}}{\partial x} & \text{on } \Gamma_{12} \Rightarrow 1^{st} \text{ order of approximation} \\ u_{(2)} = 0 & \text{on } x = 1 \end{cases}$$

+

$$\begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} U_I^{(2)\ i+1, n+1} \\ U_\Gamma^{\ i+1, n+1} \end{bmatrix} = \begin{bmatrix} f_I^{(2)} \\ f_\Gamma^{(2)} \end{bmatrix} - \begin{bmatrix} 0 \\ \phi^{i, n+1} \end{bmatrix} +$$

$$+ \begin{bmatrix} M_{II}^{(2)} & M_{I\Gamma}^{(2)} \\ M_{\Gamma I}^{(2)} & M_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} U_I^{(2)n} \\ U_\Gamma^{n} \end{bmatrix}$$

With $\phi^{i,n+1} = -f_\Gamma + A_{\Gamma\Gamma}^{(1)} U_\Gamma^{i,n+1} + A_{\Gamma I}^{(1)} U_I^{(1)n+1} - M_{\Gamma I}^{(1)} U_I^{(1)n} -$
$- M_{\Gamma\Gamma}^{2} U_\Gamma^{i,n}$

6. # Substitution scheme

The Dirichet problem remains the same as before for the 1st subdomain, with

$$u_1^{i+1} = u^{(2)\ i} \quad \text{on } \Gamma_{12}$$

However, for the Neumann problem, it changes the Neumann BC:

$$k \frac{\partial u_{(2)}^{i+1}}{\partial x} = -k \frac{\partial u_{(1)}^{i+1}}{\partial x} \quad \text{on } \Gamma_{12}$$

with:

$\phi^{i+1, n+1}$

in the 1st subdomain

With the Dirichet problem solution, it is computed the solution for the second subdomain (Neumann problem)

In iterations by subdomains scheme before advancing into new time step, the approximation is actualized with the computed one until convergence is reached.

In the Dirichet problem

$$u(1)^{i+1} = u^{i-1}|_{\Gamma} \text{ on } \Gamma_{12}$$

In the Neumann:

$$k \frac{\partial u_{(2)}^{i+1}}{\partial x} = -k \frac{\partial u_{(1)}^{i}}{\partial x}$$

7.

Recalling the weak form for the first subdomain 1 adding the contribution of the Nitche's method

$$\int w \, \partial_t u^h dx + \int_0^1 \nabla w \, k \nabla u^h dx - [w \, k \nabla u^h \underline{n}]_{\Gamma} +$$

$$+ \alpha \frac{k}{h} [w \, u^h]_{\Gamma} - k [\underline{n} \cdot \nabla w \cdot u h]_{\Gamma} =$$

$$= \int w \, f dx + \alpha \frac{k}{h} [wh, \hat{u}]_{\Gamma_{12}} - k [n \, \nabla w, \hat{u}]_{\Gamma_{12}}$$

$\hat{u} \rightarrow$ prescribed Dirichet function = considered u_2

The boundary conditions are enforced and make the method symmetric (although it subtracts stability)

α is the penalty method to ensure stability and $\hat{u}$ the prescribed Dirichlet BC at the interface for the subdomain 1.

The system to be solved is:

$$\left(\frac{M}{\delta t} + K + N\right) u_h^{n+1} = f + f_N - \frac{M}{\delta t} u_h^n$$

+ The system has to be solved for the first subdomain considering its 3 degrees of freedom, including u_0 as a Dirichlet BC.

$$u_h = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$

+ N is the matrix for the contribution of Nitsche in the LHS:

$$N = \begin{array}{cl} \begin{bmatrix} \alpha\frac{k}{h} - \frac{k}{h} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha\frac{k}{h} - \frac{k}{h} \end{bmatrix} & \begin{array}{l} \to u_0 \\ \\ \to u_2 \end{array} \end{array} \Bigg\} \text{Dirichlet BC}$$

* f_N is the contribution of Nitsche in the RHS

$$f_N = \begin{bmatrix} 0 \\ 0 \\ \left(\alpha \frac{k}{h} - k\right) \hat{u}_2 \end{bmatrix}$$

5. Operator splitting techniques

Consider 1D, transient, convection-diffusion equation:

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + a_x \frac{\partial u}{\partial x} = f \quad \text{in } [0,1]$$

$$u(x=0, t) = 0$$
$$u(x=1, t) = 0 \qquad \text{with } k = 1,\ a_x = 1,\ f = 1$$
$$u(x, t=0) = 0$$

1.

Space discretization is

$$\int_0^1 w\,(\partial_t u - k\Delta u + a_x \nabla u)\,dx = \int_0^1 w\, f\, dx$$

Integrating by parts:

$$\int_0^1 w\, k\, \Delta u\, dx = \int_0^1 \nabla w\, k \nabla u - \left[w\, k \nabla u \cdot \underline{n} \right]_0^1$$

As there is not Neumann BC, there are not any contribution to the vector force:

$$\int_0^1 w\, \partial_t u\, dx + \int_0^1 \nabla w\, k \nabla u + \int_0^1 w\, a_x \cdot \nabla u =$$

$$= \int_0^1 w\, f\, dx$$

* Using Galerkin discretization

$$u = u_h(x,t) = \sum_{i=0}^{n} u_i(t)\, N_i(x)$$

$$W^h = \text{span}\{N_j\}$$

$$\partial_t u^h(x,t) = \sum_{i=0}^{n} \partial_t u_i(t)\, N_i(x)$$

* Results to the algebraic form:

$$M \partial_t u_h + K u_h + C u_h = F$$

* Using BDF1 discretization $\partial_u t = (u^{n+1} - u^n)/\delta t$

$$\frac{M}{\delta t}\left(u_h^{n+1} - u_h^n\right) + K u_h^{n+1} + C u_h^{n+1} = F^{n+1}$$

$$\left(\frac{M}{\delta t} + K + C\right) u_h^{n+1} = F + \frac{M}{\delta t} u_h^n$$

Where the matrices are, with $h = \frac{1}{3}$

$$M_{ij} = \int_0^1 N_i N_j \, dx$$

$$\underline{M}^e = h \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

$$\underline{M_{global}} = \frac{1}{9} \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 2 & 1/2 & 0 \\ 0 & 1/2 & 2 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{bmatrix}$$

\#

$$K_{ij} = k \int \nabla N_i \nabla N_j \, dx = (-1)^{i+j} \left(\frac{k}{h}\right)^e$$

$$\underline{k}^e = \frac{1}{1/3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\underline{K_{global}} = \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

\#

$$C_{ij} = \int N_i \nabla N_j \, dx$$

$$\underline{C}^e = \frac{1}{h^2} \begin{bmatrix} -1/18 & 1/18 \\ -1/18 & 1/18 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$C_{global} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

\# $$f_i = \int N_i \cdot f \, dx$$

$$f^{(e)} = \left(\frac{f \cdot h}{2}\right)^{e1} = \left[\frac{1}{6}, \frac{1}{6}\right]$$

$$f\ global = \left[\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6}\right]$$

* Imposing Dirichlet BC $\Rightarrow [u_0, u_4 = 0]$

* IA is obtained with $u^n = [0, 0]$

$$\left(\frac{1}{9\delta t}\begin{bmatrix} 2 & 1/2 \\ 1/2 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \end{bmatrix} =$$

$$= \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} + \frac{1}{9\delta t}\begin{bmatrix} 2 & 1/2 \\ 1/2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

* Computed explicitly:

$$A \cdot \begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

* Calculated with Matlab:

$$\begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \end{bmatrix} = A^{-1}\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} =$$

$$\begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \end{bmatrix} = \frac{6\,\delta t}{2943\,\delta t^2 + 324\,\delta t + 5}\begin{bmatrix} 51\,\delta t + 1 \\ 57\,\delta t + 1 \end{bmatrix}$$

2. The operator splitting technique consists of defining

$L = L_a + L_v$

$L_a u_a = a \frac{\partial u}{\partial x}$

$L_v u_v = -k \frac{\partial^2 u}{\partial x^2}$

* The convection-diffusion is

$\partial_t u + L_a u_a + L_v u_v = f$

* First, it is solved for u_a

$u_a(t_n) = u^n$

$\partial_t u_a + L_a u_a = 0$

* Second, it is solved u_v, but the initial condition is $u_a(t_{n+1})$

$u_v(t_n) = u_a(t_{n+1})$

$\partial_t u_v + L_v u_v = f$

* Finally

$u^{n+1} = u_v(t_{n+1})$

* The discrete form is

1. Solve u_a

$$u_a^n = u^n$$

$$\left(\frac{1}{\delta t} M + C\right) u_a^{n+1} = \frac{1}{\delta t} M u_a^n$$

2. Solve u_v

$$u_v^n = u_a^{n+1}$$

$$\left(\frac{1}{\delta t} M + K\right) u_v^{n+1} = F + \frac{1}{\delta t} M u_v^n$$

3. $$u^{n+1} = u_v^{n+1}$$

* For this problem, starting with $n = 0 \Rightarrow \boxed{u^0 = 0}$

1. $$u_a^1 = \left(\frac{1}{\delta t} M + C\right)^{-1} \frac{1}{\delta t} M u_a^0 = 0$$

2. $$u_v^1 = \left(\frac{1}{\delta t} M + K\right)^{-1} F + \underbrace{\frac{1}{\delta t} M u_v^0}_{0 \; = \; u_a^1 = u_v^0}$$

3. $$\boxed{u^1 = \left(\frac{1}{\delta t} M + K\right)^{-1}}$$

* Using same matrices, it is computed the solution with Matlab

$$u^1 = \begin{bmatrix} 0 \\ \dfrac{6\,\delta t}{54\,\delta t + 5} \\ \dfrac{6\,\delta t}{54\,\delta t + 5} \\ 0 \end{bmatrix}$$

3.

The absolute error of the operator-splitting is computed as:

$$e = \left| \underline{U}_{reduced,\ monolithic} - U_{reduced,\ splitting} \right|$$

The solutions for $\delta t = 1$, $\delta t = 0.5$, $\delta t = 0.25$ and their error are shown next.

3. Evaluate the error of the splitting approach with respect to the monolithic approach. Plot the splitting error vs. the time step size for Δt = 1; Δt = 0:5, Δt = 0:25: Comment on the results.

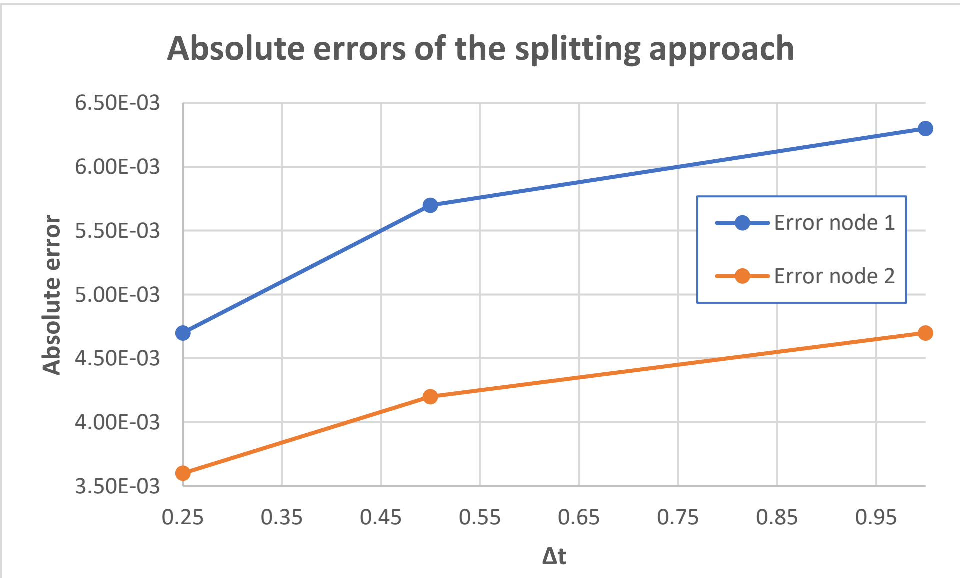
Next, it is presented the solutions for each approach.

	Solutions					
	Splitting operator			Monolithic		
Δt	*1*	*0.5*	*0.25*	*1*	*0.5*	*0.25*
Node 1	1.02E-01	9.38E-02	8.11E-02	9.54E-02	8.81E-02	7.64E-02
Node 2	1.02E-01	9.38E-02	8.11E-02	1.06E-01	9.80E-02	8.47E-02

Next, it is presented the absolute error of the splitting approach with respect to the monolithic approach.

	Absolute error		
	Splitting operator		
Δt	*1*	*0.5*	*0.25*
Error node 1	6.30E-03	5.70E-03	4.70E-03
Error node 2	4.70E-03	4.20E-03	3.60E-03

Finally, it is plotted the splitting error vs. the time step.



6. Fractional steps methods

Consider the fractional step approach for the incompressible Navier-Stokes equation (Yosida scheme)

1. Optimal value of the α parameter

To find out the optimal value of α, first it written the discrete form of the incompressible Navier-Stokes equations using BDF1 time integration without decoupling of pressure and velocity:

$$M \frac{1}{\delta t} (U^{n+1} - U^n) + KU^{n+1} = f - G P^{n+1}$$

$$DU^{n+1} = 0$$

Summing the 1^{st} and 3^{rd} equation of Yosida scheme, yields:

$$M \frac{1}{\delta t} (U^{n+1} - U^n) + K(\hat{U}^{n+1} + \alpha U^{n+1} - \alpha \hat{U}^{n+1}) +$$

$$+ \theta (U^{n+1} - \hat{U}^{n+1}) = f - G P^{n+1}$$

Where it is concluded that if $[\alpha = 1]$, it is recovered the momentum equation of Navier-Stokes. Therefore, $\alpha = 1$ is the optimal value

2

As it is shown before, with $\alpha=1$, it is obtained an error of order $O(U^{n+1}-\hat{U}^{n+1})$ as the convection term is non-linear.
The biggest error is associated with the imposition of the incompressibility condition.
Firstly, in the Yanida scheme is calculated $\hat{U}^{n+1}$ as the velocity that satisfies only the momentum equation.
Secondly, in the second equation which enforces the incompressibility, it is calculated the aproximation of the gradient of the pressure to assure the continuity $(P^{n+1}-\hat{P}^{n+1})$
Thirdly in the third equation, U^{n+1} is calculated as a correction of $\hat{U}^{n+1}$ with a better approximation of the condition of continuity.

7. ALE formulations

1. Spatial description of a property

$$\Upsilon(x,y,z,t) = [2x, ye^t, z]$$

Equations of movement:

$$x = X \cdot e^t$$
$$y = Y + e^t - 1$$
$$z = Z$$

Equations of the movement of the mesh:

$$x_m = \hat{X} + \alpha t$$
$$y_m = \hat{Y} - \beta t$$
$$z_m = \hat{Z}$$

a) The mapping between the mesh nodes $\underline{\hat{X}}$ and the spatial coordinates is:

$$\underline{x} = \phi(\underline{\hat{X}}, t) = \underline{x}(\underline{\hat{X}}, t)$$

The ALE coordinates are:

$$\Upsilon_{ALE}(\hat{X}, \hat{Y}, \hat{Z}, t) = \begin{bmatrix} 2(\hat{X} + \alpha t) \\ (\hat{Y} - \beta t)e^t \\ \hat{Z} \end{bmatrix}$$

$$\Upsilon_{ALE} = \Upsilon(\underline{x}, t) = \Upsilon(\phi(\underline{\hat{X}}, t))$$

b) The velocity of the particle is

$$\underline{v} = \frac{\partial \underline{x}(\underline{X},t)}{\partial t} = \begin{bmatrix} Xe^t \\ e^t \\ 0 \end{bmatrix}$$

The velocity of the mesh is:

$$\underline{v_{mesh}} = \frac{\partial x(\hat{\underline{X}},t)}{\partial t} = \begin{bmatrix} \alpha \\ -\beta \\ 0 \end{bmatrix}$$

c) The material temporal derivative of $\gamma_{ALE}(\hat{\underline{X}},t)$ is:

$$\frac{d}{dt}\gamma_{ALE}(\hat{\underline{X}},t) = \frac{\partial \gamma_{ALE}(\hat{\underline{X}},t)}{\partial t} + \nabla\gamma(x,t)\cdot(\underline{v}-\underline{v_{mesh}})$$

$$\nabla\gamma(x,t) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial \gamma_{ALE}}{\partial t} = \begin{bmatrix} 2\alpha \\ (\hat{Y}-\beta(1+t))e^t \\ 0 \end{bmatrix}$$

$$\underline{v}-\underline{v_{mesh}} = \begin{bmatrix} Xe^t-\alpha \\ e^t+\beta \\ 0 \end{bmatrix};$$

$$\frac{d}{dt} v_{ALE}(\hat{\underline{X}}, t) = \begin{bmatrix} 2Xe^t \\ (\hat{Y} - \beta t + e^t)e^t \\ 0 \end{bmatrix}$$

* From eq. of movement:

$$x = Xe^t = \hat{X} + \alpha t \Rightarrow X = (\hat{X} + \alpha t)e^{-t}$$

* This yields to:

$$\frac{d}{dt} v_{ALE}(\hat{\underline{X}}, t) = \begin{bmatrix} 2(\hat{X} + \alpha t) \\ (\hat{Y} - \beta t + e^t)e^t \\ 0 \end{bmatrix}$$

2.

* The momentum conservation equation in ALE is:

$$\frac{\partial u_{ALE}(\hat{\underline{X}}, t)}{\partial t} + \underline{c} \cdot \underline{\nabla} \underline{u}(x,t) - \underline{\nabla} \cdot \underline{\sigma}(x,t) = \rho(x,t)\, \underline{b}(x,t)$$

where c is $\underline{c} = \underline{v} - \underline{v}_{mesh}$

* The Cauchy stress tensor $\sigma(x,t)$

$$\sigma(x,t) = -p(x,t)\underline{I} + 2\mu \nabla^S u(x,t)$$

*

$$\frac{\partial u_{ALE}(\hat{X}, t)}{\partial t} + c \cdot \nabla u(x,t) + \nabla p(x,t) - \mu \nabla^2 u(x,t) =$$

$$= \rho(x,t)\, \underline{b}(x,t)$$

* The mass balance is:

$$\frac{\partial \rho_{ALE}(\hat{\underline{X}},t)}{\partial t} + c \cdot \nabla \rho(x,t) = -\rho\, \underline{\nabla} \cdot \underline{u}(x,t)$$

* For a incompressible flow is simplified as:

$$\underline{\nabla} \cdot \underline{u}(x,t) = 0$$

* The ALE form of incompressible Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial u_{ALE}(\hat{\underline{X}},t)}{\partial t} + \underline{c} \cdot \underline{\nabla}\underline{u}(x,t) + \nabla p(\underline{x},t) - \mu \nabla^2 u(x,t) = \\ \qquad = \rho(x,t) \cdot b(x,t) \\ \nabla \cdot u(x,t) = 0 \end{cases}$$

Most of terms are evaluated in the Eulerian frame of referance.
Regarding the temporal derivatives, u_{ALE} is evaluated at the same nodes even if they move. This is due to the fact that the temporal derivatives are discretized using finite difference method.

3.

First, it is stated the requirements that mesh movement methods have to justify:

- In the interior of the domain, the mesh movement has to avoid excessively distorted elements so there is not an extra numerical error associated
- In some boundaries of the domain, it has to stay static (Eulerian boundary) $\Rightarrow d_{mesh} = 0 \; \Gamma_{Eulerian}$
- In some boundaries of the domain, it has to go with the movement of the particles in the boundaries (lagrangian boundary) $\Rightarrow d_{mesh} = d_L$ on $\Gamma_{lagrangian}$.

Therefore, the problem is that given the displacement of the mesh in the boundary, it is required to find a map that fullfils these boundary conditions and avoids this excessive distorsion in the interior mesh. The following techniques are used:

- Mesh-smoothing: it is used a mesh-smoothing algorithm if the topology is conserved, assuring a reduction of the deformation of the mesh shape
- laplacian mesh: it is enforced that each component of the displacement is harmonic:

$$\nabla^2 d_i \quad \text{for } i = 1, \dots, nd$$

The drawback is that it may produce self-interaction for big displacements.

- Elasticity problem: it is considered that the domain behaves as an elastic body. It is provided the displacements on the boundary, then the displacements produced in the interior are solved as a linear elasticity problem. This method is more accurate than Laplacian technique but it requires a higher computational cost. This is due to the fact the displacements in different directions are coupled.

- Transfinite Mapping: it is described an approximate surface or volume at a huge number of points with specified boundaries. For a 2D case, the method can model exactly all domain boundaries; therefore, there is not any error introduced by the mapping. It does not require much computational cost due to the fact the boundaries have been discretized, so the nodal coordinates are obtained explicitly

Referances:

[1] Arbitrary Lagrangian-Eulerian Methods. J. Donea, A. Huerta

[2] Arbitrary Lagrangian-Eulerian formulation for hyperelasticity. Rodriguez Ferran

8. Fluid - Structure Interaction

1\.

In Fluid-Structure Interaction, the problem of added mass effect appears when the densities of the incompressible fluid and the solid object are similar.
This implies that it is not reached the convergance in partitioned schemes.
In order to fix this issue of the partitioned schemes, it is used methods which use relaxation parameters.
These methods reduce the instabilities weightening the Dirichlet condition applied at the interface of one of the sub-domains.
One method used to solve this problem is aitken relaxation scheme which takes into account the last two iterations to approximate the next one.

2\. The heat transfer problem in 1D of Section 4 is:

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f \quad \text{in } [0, 1]$$

$$\begin{cases} u(x=0, t) = 0 \\ u(x=1, t) = 0 \\ u(x, t=0) = 0 \end{cases}$$

* As it is shown in problem 1 of Section 4, the problem is discretized using BDF1:

$$\delta_t U^{n+1} = (U^{n+1} - U^n)/\delta t$$ (unconditionally stable => $O(\delta t)$ implicit scheme

$$\left(\frac{1}{\delta t}\underline{M} + \underline{K}\right)\underline{u}^{n+1} = \underline{f} + \frac{1}{\delta t}\underline{M}\,\underline{u}^n$$

* Contribution of Neumann Boundary condition at the interface to solve first subdomain:

$$\left(\frac{1}{\delta t} M_1 + K_1\right) u_1^{n+1,k} = f_1 + \frac{1}{\delta t} M_1 u_1^n$$

where f_1 is the Neumann BC contribution:

$$\begin{cases} \dfrac{\partial u_1^{n+1,k}}{\partial x}(\Gamma_{12}) = -\dfrac{\partial u_2^{n+1,k-1}}{\partial x}(\Gamma_{12}) & \text{on } \Gamma_{12} \\ u_1^{n+1,k} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma_{12} \end{cases}$$

* Dirichet problem in the second subdomain

$$\left(\frac{1}{\delta t} M_2 + K_2\right) u_2^{*\,n+1,k} = f_2 + \frac{1}{\delta t} M_2 u_2^n$$

* It is used Dirichet - Neumann algorithm

- Using Aitken scheme on the Γ_{12}, and $u_2^{n+1} = 0$ on $\partial\Omega_2 \backslash \Gamma_{21}$

$$u_2^{n+1,k}(\Gamma_{12}) = \omega\, u_1^{n+1,k}(\Gamma_{12}) + (1-\omega)\, u_2^{*\,n+1,k-1}(\Gamma_{12})$$

Where the relaxation factor (ω) is:

$$\omega = \frac{u_2^{n+1,k-2}(\Gamma) - u_2^{n+1,k-1}(\Gamma)}{u_2^{n+1,k-2}(\Gamma) - u_2^{n+1,k-1}(\Gamma) + u_2^{*\,n+1,k}(\Gamma) - u_2^{*\,n+1,k-1}(\Gamma)}$$

A value for "ω" has to be selected due to the fact this scheme is not valid for the two first iterations. The two first iterations of the Aitken scheme are the same as the partitioned scheme.

3. Considering monolithic scheme, transient (BDF1) and linear elements with $h = 1/4 \Rightarrow 4$ elements in $[0,1]$

- The element stiffness is: assuming $k=1$

$$K^e = \frac{K}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

- The mass matrix is:

$$M^e = h\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/12 & 1/24 \\ 1/24 & 1/12 \end{bmatrix}$$

The asembling yields to:

$$K = \begin{bmatrix} 4 & -4 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}$$

$$M = \frac{1}{24} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The Dirichet boundary conditions are enforced using lagrange multipliers: (in $x=0$ and $x=1$)

$L_1(0) = 1 \qquad L_1(1) = 0$

$L_2(0) = 0 \qquad L_2(1) = 1$

Resulting to: $\rightarrow$ 5 nodes for 4 elements

$$Lu = b$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

* Therefore, the discretized system of the global problem:

$$\begin{bmatrix} A & L^T \\ L & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ \lambda \end{bmatrix} = \begin{bmatrix} \hat{f} \\ b \end{bmatrix}$$

* Where the global problem is stated as:

$$\underline{A} \; u^{n+1} = \hat{f} \quad \begin{cases} \underline{A} = \dfrac{M}{\delta t} + \underline{k} \\ \hat{f} = f + \dfrac{1}{\delta t} M u^{n} \end{cases}$$

* Assuming $\delta t = 1$ and $\hat{A} = \begin{bmatrix} A & L^T \\ L & 0 \end{bmatrix}$

* The condition number of $\hat{A}$ is:

$$k_{\hat{A}} = \|\underline{A}\| \cdot \|\underline{\underline{A}}^{-1}\| = 38.3156$$ (computed in Matlab)

* Whereas, the condition number of the original matrix A is:

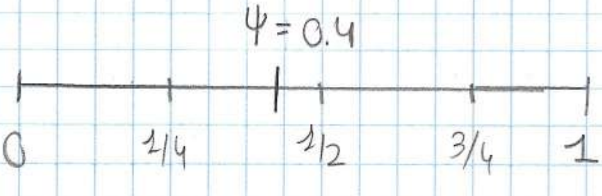
$$k_A = 73.0412$$

* Therefore, despite the fact it is introduced more equations with the lagrange multipliers, the condition number is reduced almost by one half and it is saved computational cost.

4. Monolithic $\Rightarrow h = 1/4$

Subdomain 1 $\Rightarrow [0, 0.4]$ with $k_1 = 100$

Subdomain 2 $\Rightarrow [0.4, 1]$ with $k_2 = 1$



* The elemental mass matrices do not change, therefore, the global mass matrix is still the same.

* As for the elemental matrices:

$$K^1 = 100 \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \qquad K^3 = K^4 = 1 \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

* For the second element:

$$K^2 = \int_{1/4}^{1/2} \begin{bmatrix} \nabla N_1^e \, k \, \nabla N_1^e & \nabla N_1^e \, k \, \nabla N_2^e \\ \nabla N_2^e \, k \, \nabla N_1^e & \nabla N_2^e \, k \, \nabla N_2^e \end{bmatrix} dx =$$

$$= \int_{0.25}^{0.4} 100 \cdot \begin{bmatrix} \nabla N_1^e \, \nabla N_1^e & \nabla N_1^e \, \nabla N_2^e \\ \nabla N_2^e \, \nabla N_1^e & \nabla N_2^e \, \nabla N_2^e \end{bmatrix} dx +$$

$$+ 1 \int_{0.4}^{0.5} \begin{bmatrix} \nabla N_1^e \, \nabla N_1^e & \nabla N_1^e \, \nabla N_2^e \\ \nabla N_2^e \, \nabla N_1^e & \nabla N_2^e \, \nabla N_2^e \end{bmatrix} dx =$$

$$= 100 \begin{bmatrix} 4 \cdot 4 & 4 \cdot (-4) \\ (-4) \cdot 4 & (-4) \cdot (-4) \end{bmatrix} (0.4 - 0.25) +$$

$$+ \begin{bmatrix} 4 \cdot 4 & 4 \cdot (-4) \\ (-4) \cdot 4 & (-4) \cdot (-4) \end{bmatrix} \cdot (0.5 - 0.4) \Rightarrow$$

$$k^2 = 240.16 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

* The global stiffness matrix is:

$$k = \begin{bmatrix} 400 & -400 & 0 & 0 & 0 \\ -400 & 640.16 & -240.16 & 0 & 0 \\ 0 & -240.16 & 244.16 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}$$