COUPLED PROBLEMS

Master of Science in Computational Mechanics/Numerical Methods

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Homework

1. Transmission conditions

1.1. The deflection $v(x)$ of an Euler-Bernoulli beam is governed by the differential equation

$$EI\frac{d^4v}{dx^4} = f$$

Where EI is a mechanical property of the beam section and the beam material and f is the distributed load. Assuming for example that the beam is clamped at $x = 0$ and $x = L$, the Principle of Virtual Work (PTV) states that the solution $v(x)$ satisfies

$$EI\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx = \int_0^L \delta v f\,dx$$

For all δv such that $\delta v(0) = \delta v(L) = 0, \frac{d\delta v}{dx}(0) = \frac{d\delta v}{dx}(L) = 0.$

a) **Postulate the space of functions where both v and δv must belong. Justify the answer**.

Both trial and test functions must satisfy the following condition: Their second derivative must be square integrable in the domain of the problem. This ensures $\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx < \infty$. The space of functions that fulfil this condition is, by definition $W_2^2([0,L]) = H^2([0,L])$. Virtual displacements must satisfy additional conditions: their value and the value of their derivative must vanish at the extremes of the domain. So δv belong in a subspace of $H^2([0,L])$: $\delta v \in H_0^2([0,L]) = \left\{w \in H^2([0,L]): w(0) = w(L) = 0, \frac{dw}{dx}(0) = \frac{dw}{dx}(L) = 0\right\}$.

Moreover, the function f must ensure $\int_0^L \delta v f\,dx < \infty$. This means that f must belong to the dual space of δv. $f \in (W_2^2)' = W_2^{-2}$.

b) **If** $[0,L] = [0,P] \cup (P,L]$, **obtain the transmission conditions at P implied by regularity requirements.**

As $v \in H^2([0,L])$ it can be shown that v must be continuous and also its derivative in all the domain.

Sobolev's Inequality states that for $p > 1$ a function in W_p^k is C^m where

$$k - m > n/p$$

here $k = 2$ and $p = 2$ are the parameters of the Sobolev space, $n = 1$ is the number of dimensions of the domain. In this case, $m < 2 - \frac{1}{2} = \frac{3}{2} \rightarrow m = 1$. This means that $u \in C^1$.

At point P this is stated as $[\![v]\!] = 0$ and $\left[\!\!\left[\frac{dv}{dx}\right]\!\!\right] = 0$.

c) **Obtain the transmission conditions at P that follow by imposing in the PTV that the integral is additive.**

From the strong form of the differential equation:

$$EI\frac{d^4v}{dx^4} = f \rightarrow \int_0^L \delta v f \, dx = \int_0^L EI\,\delta v\frac{d^4v}{dx^4}\,dx = \int_0^P EI\,\delta v\frac{d^4v}{dx^4}\,dx + \int_P^L EI\,\delta v\frac{d^4v}{dx^4}\,dx$$

Integrating the second term by parts twice:

$$\int_0^P EI\,\delta v\frac{d^4v}{dx^4}\,dx + \int_P^L EI\,\delta v\frac{d^4v}{dx^4}\,dx$$
$$= -\int_0^P EI\,\frac{d\delta v}{dx}\frac{d^3v}{dx^3}\,dx + \left[EI\,\delta v\frac{d^3v}{dx^3}\right]_0^P - \int_P^L EI\,\frac{d\delta v}{dx}\frac{d^3v}{dx^3}\,dx + \left[EI\,\delta v\frac{d^3v}{dx^3}\right]_P^L =$$
$$\int_0^P EI\,\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx + \left[EI\,\delta v\frac{d^3v}{dx^3}\right]_0^P - \left[EI\,\frac{d\delta v}{dx}\frac{d^2v}{dx^2}\right]_0^P + \int_P^L EI\,\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx + \left[EI\,\delta v\frac{d^3v}{dx^3}\right]_P^L$$
$$- \left[EI\,\frac{d\delta v}{dx}\frac{d^2v}{dx^2}\right]_P^L$$

Now it is used the property of the virtual displacement function $\delta v(0) = \delta v(L) = 0, \frac{d\delta v}{dx}(0) = \frac{d\delta v}{dx}(L) = 0$:

$$\int_0^L \delta v f\,dx = \int_0^P EI\,\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx + \int_P^L EI\,\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx + EI\,\delta v(P^-)\frac{d^3v(P^-)}{dx^3}$$
$$- EI\,\frac{d\delta v(P^-)}{dx}\frac{d^2v(P^-)}{dx^2} - EI\,\delta v(P^+)\frac{d^3v(P^+)}{dx^3} + EI\,\frac{d\delta v(P^+)}{dx}\frac{d^2v(P^+)}{dx^2}$$

Imposing that the integral of PTV formulation is additive:

$$\int_0^L \delta v f\,dx = \int_0^L EI\,\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx = \int_0^P EI\,\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx + \int_P^L EI\,\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx$$

Comparing the two equations:

$$\int_0^P EI\,\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx + \int_P^L EI\,\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx + EI\,\delta v(P^-)\frac{d^3v(P^-)}{dx^3} - EI\,\frac{d\delta v(P^-)}{dx}\frac{d^2v(P^-)}{dx^2}$$
$$- EI\,\delta v(P^+)\frac{d^3v(P^+)}{dx^3} + EI\,\frac{d\delta v(P^+)}{dx}\frac{d^2v(P^+)}{dx^2}$$
$$= \int_0^P EI\,\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx + \int_P^L EI\,\frac{d^2\delta v}{dx^2}\frac{d^2v}{dx^2}\,dx$$

Resulting in:

$$EI\,\delta v(P^-)\frac{d^3v(P^-)}{dx^3} - EI\,\frac{d\delta v(P^-)}{dx}\frac{d^2v(P^-)}{dx^2} - EI\,\delta v(P^+)\frac{d^3v(P^+)}{dx^3} + EI\,\frac{d\delta v(P^+)}{dx}\frac{d^2v(P^+)}{dx^2} = 0$$

Taking into account that δv is arbitrary:

$$EI\,\delta v(P^-)\frac{d^3v(P^-)}{dx^3} - EI\,\delta v(P^+)\frac{d^3v(P^+)}{dx^3} = 0 \rightarrow \begin{cases} \delta v(P^-) = \delta v(P^+) \\ EI\dfrac{d^3v(P^-)}{dx^3} - EI\dfrac{d^3v(P^+)}{dx^3} = 0 \end{cases}$$

$$-EI\,\frac{d\delta v(P^-)}{dx}\frac{d^2v(P^-)}{dx^2} + EI\,\frac{d\delta v(P^+)}{dx}\frac{d^2v(P^+)}{dx^2} = 0 \rightarrow \begin{cases} \dfrac{d\delta v(P^-)}{dx} = \dfrac{d\delta v(P^+)}{dx} \\ -EI\dfrac{d^2v(P^-)}{dx^2} + EI\dfrac{d^2v(P^+)}{dx^2} = 0 \end{cases}$$

It is seen that 4 transmission conditions are obtained:

$$[\![\delta v]\!] = 0 \; and \; \left[\!\!\left[\frac{d\delta v}{dx}\right]\!\!\right] = 0$$

$$\left[\!\!\left[EI\frac{d^2v}{dx^2}\right]\!\!\right] = 0 \; and \; \left[\!\!\left[EI\frac{d^3v}{dx^3}\right]\!\!\right] = 0$$

The transmission conditions of the virtual displacements are redundant as $\delta v \in H^2([0, L])$.

It has to be noted that all of these conditions are stated in terms of class equivalence. That is, they can be not fulfilled in sets of 0 measure.

1.2. The Maxwell problem consists in finding a vector field $u: \Omega \rightarrow \mathbb{R}^3$ such that

$$\nu\nabla \times \nabla \times \boldsymbol{u} = f \; in \; \Omega$$
$$\nabla \cdot \boldsymbol{u} = 0 \; in \; \Omega$$
$$\boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{0} \; on \; \partial\Omega$$

Where $\nu > 0, \boldsymbol{f}$ is a divergence free force field and $\boldsymbol{n}$ the unit external normal. Equation $\nabla \cdot u = 0$ is in fact redundant.

a) Write a variational statement of the problem. Postulate the space of functions where $\boldsymbol{u}$ must belong. Justify the answer.

In this formulation of the strong form of the PDE it has been assumed ν is constant along the domain. For a more general formulation is taken

$$\nabla \times (\nu\nabla \times \boldsymbol{u}) = \boldsymbol{f} \; in \; \Omega$$

The variational formulation is obtained integrating the equation multiplied by a test function $\delta\boldsymbol{u} \in \mathbb{R}^3(\Omega)$ such that $\delta\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \; on \; \partial\Omega$ in order to apply the Dirichlet BC:

$$\int_\Omega \delta\boldsymbol{u} \cdot \nabla \times (\nu\nabla \times \boldsymbol{u})\, d\Omega = \int_\Omega \delta\boldsymbol{u} \cdot \boldsymbol{f} d\Omega$$

The following formula is used:

$$\boldsymbol{u} \cdot (\nabla \times \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \times \boldsymbol{u} - \nabla \cdot (\boldsymbol{u} \times \boldsymbol{v})$$

substituting $\boldsymbol{u}$ by $\delta\boldsymbol{u}$ and $\boldsymbol{v}$ by $(\nu\nabla \times \boldsymbol{u})$:

$$\begin{aligned}\int_\Omega \delta\boldsymbol{u} \cdot \nabla \times (\nu\nabla \times \boldsymbol{u})\, d\Omega &= \int_\Omega \nabla \times \delta\boldsymbol{u} \cdot (\nu\nabla \times \boldsymbol{u})\, d\Omega - \int_\Omega \nabla \cdot \big(\delta\boldsymbol{u} \times (\nu\nabla \times \boldsymbol{u})\big) d\Omega \\ &= \int_\Omega \nabla \times \delta\boldsymbol{u} \cdot (\nu\nabla \times \boldsymbol{u})\, d\Omega - \int_{\partial\Omega} \delta\boldsymbol{u} \times (\nu\nabla \times \boldsymbol{u}) \cdot \boldsymbol{n} d\Gamma \\ &= \int_\Omega \nabla \times \delta\boldsymbol{u} \cdot (\nu\nabla \times \boldsymbol{u})\, d\Omega + \int_{\partial\Omega} \delta\boldsymbol{u} \times \boldsymbol{n} \cdot (\nu\nabla \times \boldsymbol{u}) d\Gamma = \int_\Omega \nabla \times \delta\boldsymbol{u} \cdot (\nu\nabla \times \boldsymbol{u})\, d\Omega\end{aligned}$$

To ensure that the integral is bounded the term $\nabla \times \delta \boldsymbol{u} \cdot (\nu \nabla \times \boldsymbol{u})$ must be in $L_1(\Omega)$. As this term is the product of two terms this means that each of the terms must be in $L_2(\Omega)$:

$$\nabla \times \delta \boldsymbol{u} \in L_2(\Omega) = H^0(\Omega) \rightarrow \delta \boldsymbol{u} \in H^0(curl, \Omega) = \{\boldsymbol{u} \in L_2(\Omega) : \nabla \times \boldsymbol{u} \in L_2(\Omega)\}$$

As it is desired that the obtained weak form is symmetric:

$$\boldsymbol{u} \in H^0(curl, \Omega)$$

This ensures $\nabla \times \boldsymbol{u} \in L_2(\Omega)$. As $\nu(\nabla \times \boldsymbol{u}) \in L_2(\Omega)$ this means that $\nu \in L_\infty(\Omega)$. Vector of force $\boldsymbol{f} \in \left(H^0(curl, \Omega)\right)'$, the dual space which belongs $\delta \boldsymbol{u}$.

The additional condition $\delta \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ can be applied stating that $\delta \boldsymbol{u} \in H_0^0(curl, \Omega) = \{\boldsymbol{u} \in H^0(curl, \Omega) : \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \ on \ \partial\Omega\}$

b) If Γ is a surface that intersects Ω, obtain the transmission conditions across Γ implied by regularity requirements.

As $\boldsymbol{u} \in H^0(curl, \Omega) \rightarrow \nabla \times \boldsymbol{u} \in L_2(\Omega)$. The *i*-th component of $\nabla \times \boldsymbol{u}$ is:

$$[\nabla \times \boldsymbol{u}]_i = \epsilon_{ijk} u_{k,j}$$

This is written as $u_{k,j} \in L_2(\Omega) \ for \ k \neq j$.

Now imagine that u_k is discontinuous along the *j* dimension at x_0. A regularized function u^ϵ is defined as a continuous function such that

$$u^\epsilon(x) = u_k(x) \ for \ |x - x_0| > \frac{\epsilon}{2} \text{ and}$$

$$u^\epsilon(\boldsymbol{x}) = \frac{u\left(x_0 + \frac{\epsilon}{2}\right) + u\left(x_0 - \frac{\epsilon}{2}\right)}{2} + \frac{u\left(x_0 + \frac{\epsilon}{2}\right) - u\left(x_0 - \frac{\epsilon}{2}\right)}{\epsilon}(x - x_0)$$

With this definition $u^\epsilon \rightarrow u$ when $\epsilon \rightarrow 0$.

The square integral of $u_{k,j}$ around an interval of length $2a$ centered at x_0 is:

$$\int_{x_0-a}^{x_0+a} \left(\frac{du_k}{dx_j}\right)^2 dx_j = \lim_{\epsilon \to 0} \int_{x_0-a}^{x_0-\frac{\epsilon}{2}} \left(\frac{du^\epsilon}{dx_j}\right)^2 dx_j + \int_{x_0+\frac{\epsilon}{2}}^{x_0+a} \left(\frac{du^\epsilon}{dx_j}\right)^2 dx_j + \int_{x_0-\frac{\epsilon}{2}}^{x_0+\frac{\epsilon}{2}} \left(\frac{du^\epsilon}{dx_j}\right)^2 dx_j =$$

$$\lim_{\epsilon \to 0} \int_{x_0-a}^{x_0-\frac{\epsilon}{2}} \left(\frac{du^\epsilon}{dx_j}\right)^2 dx_j + \int_{x_0+\frac{\epsilon}{2}}^{x_0+a} \left(\frac{du^\epsilon}{dx_j}\right)^2 dx_j + \epsilon \left[\frac{u\left(x_0 + \frac{\epsilon}{2}\right) - u\left(x_0 - \frac{\epsilon}{2}\right)}{\epsilon}\right]^2$$

This tends to ∞ as $\epsilon \rightarrow 0$. That means that tangential components of u must be continuous along surfaces (excepts on sets of measure 0).

c) Obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive.

From the strong form of the PDE:

$$\nu\nabla \times \nabla \times \boldsymbol{u} = \boldsymbol{f} \rightarrow \int_{\Omega} \delta\boldsymbol{u}\cdot\boldsymbol{f}d\Omega = \int_{\Omega} \delta\boldsymbol{u}\cdot(\nu\nabla\times\nabla\times\boldsymbol{u})d\Omega$$
$$= \int_{\Omega_1} \delta\boldsymbol{u}\cdot(\nu\nabla\times\nabla\times\boldsymbol{u})d\Omega + \int_{\Omega_2} \delta\boldsymbol{u}\cdot(\nu\nabla\times\nabla\times\boldsymbol{u})d\Omega$$

Where $\Omega_1 \cup \Omega_2 = \Omega$ and $\Omega_1 \cap \Omega_2 = \Gamma$.

Applying to both terms the following formula of vector calculus and the divergence theorem as in a):

$$\boldsymbol{u}\cdot(\nabla\times\boldsymbol{v}) = \boldsymbol{v}\cdot\nabla\times\boldsymbol{u} - \nabla\cdot(\boldsymbol{u}\times\boldsymbol{v})$$

It is obtained

$$\int_{\Omega_1} \delta\boldsymbol{u}\cdot(\nu\nabla\times\nabla\times\boldsymbol{u})d\Omega + \int_{\Omega_2} \delta\boldsymbol{u}\cdot(\nu\nabla\times\nabla\times\boldsymbol{u})d\Omega$$
$$= \int_{\Omega_1} \nabla\times\delta\boldsymbol{u}\cdot(\nu\nabla\times\boldsymbol{u})\,d\Omega + \int_{\Gamma} \delta\boldsymbol{u}\times\left(\nu\nabla\times\boldsymbol{u}|_{\Omega_1}\right)\cdot\boldsymbol{n}d\Gamma + \int_{\Omega} \nabla\times\delta\boldsymbol{u}\cdot(\nu\nabla\times\boldsymbol{u})\,d\Omega$$
$$+ \int_{\Gamma} \delta\boldsymbol{u}\times\left(\nu\nabla\times\boldsymbol{u}|_{\Omega_2}\right)\cdot\boldsymbol{n}d\Gamma$$

Where it has been used that $\delta\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}\ on\ \partial\Omega$. From the weak form:

$$\int_{\Omega} \delta\boldsymbol{u}\cdot\boldsymbol{f}d\Omega = \int_{\Omega} \nabla\times\delta\boldsymbol{u}\cdot(\nu\nabla\times\boldsymbol{u})\,d\Omega = \int_{\Omega_1} \nabla\times\delta\boldsymbol{u}\cdot(\nu\nabla\times\boldsymbol{u})\,d\Omega + \int_{\Omega_2} \nabla\times\delta\boldsymbol{u}\cdot(\nu\nabla\times\boldsymbol{u})\,d\Omega$$

As $\delta\boldsymbol{u}$ is arbitrary:

$$\int_{\Gamma} \delta\boldsymbol{u}\times\left(\nu\nabla\times\boldsymbol{u}|_{\Omega_1}\right)\cdot\boldsymbol{n}\,d\Gamma + \int_{\Gamma} \delta\boldsymbol{u}\times\boldsymbol{n}\cdot\left(\nu\nabla\times\boldsymbol{u}|_{\Omega_2}\right)\cdot\boldsymbol{n}\,d\Gamma = 0$$

This means that:

$$[\![\nu(\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}]\!] = 0$$

1.3. The Navier equations for an elastic material can be written in three different ways:

$$-2\mu\nabla\cdot\left(\varepsilon(u)\right) - \lambda\nabla(\nabla\cdot u) = \rho b$$
$$-\mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u) = \rho b$$
$$\mu\nabla\times(\nabla\times u) - (\lambda+2\mu)\nabla(\nabla\cdot u) = \rho b$$

Where u is the displacement field, $\varepsilon(u)$ the symmetric part of ∇u,λ and μ the Lamé coefficients, ρ the density of the material and $\boldsymbol{b}$ the body force. Let us assume that $u = 0$ on $\partial\Omega$.

a) Write down the variational form of the previous equation in the appropriate functional spaces.

- **1st formulation**

$$\int_{\Omega} \delta u\cdot\left(-2\mu\nabla\cdot\left(\varepsilon(u)\right) - \lambda\nabla(\nabla\cdot u)\right)\,d\Omega = \int_{\Omega} \delta u\cdot\rho b\,d\Omega$$

The left hand side can be integrated by parts:

$$\int_\Omega \delta u \cdot \left(-2\mu\nabla \cdot (\varepsilon(u)) - \lambda\nabla(\nabla \cdot u)\right)\, d\Omega$$

$$= \int_\Omega \nabla\delta u : 2\mu\big(\varepsilon(u)\big)\, d\Omega - \int_{\partial\Omega} \delta u \cdot\ 2\mu\varepsilon(u) \cdot \boldsymbol{n}\, d\Gamma + \int_\Omega \nabla\delta u \cdot\ \lambda(\nabla \cdot u)\, d\Omega$$

$$- \int_{\partial\Omega} \delta u \cdot\ \lambda(\nabla \cdot u) \cdot \boldsymbol{n}\, d\Gamma$$

Resulting in

$$\int_\Omega \nabla\delta u : 2\mu\big(\varepsilon(u)\big)\, d\Omega - \int_{\partial\Omega} \delta u \cdot\ 2\mu\varepsilon(u) \cdot \boldsymbol{n}\, d\Gamma + \int_\Omega \nabla\delta u \cdot\ \lambda(\nabla \cdot u)\, d\Omega - \int_{\partial\Omega} \delta u \cdot\ \lambda(\nabla \cdot u) \cdot \boldsymbol{n} d\Gamma$$
$$= \int_\Omega \delta u \cdot \rho b\, d\Omega$$

Here the most restrictive term is

$$\int_\Omega \nabla\delta u : 2\mu\big(\varepsilon(u)\big)\, d\Omega$$

This is because it uses the full gradient of the vector unknown. This means that $u, \delta u \in H^1(\Omega)$

- **2nd formulation**

$$\int_\Omega \delta u \cdot \big(-\mu\Delta u - (\lambda + \mu)\nabla(\nabla \cdot u)\big) d\Omega = \int_\Omega \delta u \cdot \rho b\, d\Omega$$

The left hand side is integrated by parts:

$$\int_\Omega \delta u \cdot \big(-\mu\Delta u - (\lambda + \mu)\nabla(\nabla \cdot u)\big) d\Omega$$

$$= \int_\Omega \nabla\delta u \cdot \mu\nabla u\, d\Omega - \int_{\partial\Omega} \delta u \cdot \mu\nabla u \cdot \boldsymbol{n} d\Gamma + \int_\Omega \nabla\delta u \cdot (\lambda + \mu)(\nabla \cdot u)\, d\Omega$$

$$- \int_{\partial\Omega} \delta u \cdot (\lambda + \mu)(\nabla \cdot u) \cdot \boldsymbol{n}\, d\Gamma$$

The weak formulation is:

$$\int_\Omega \nabla\delta u \cdot \mu\nabla u\, d\Omega - \int_{\partial\Omega} \delta u \cdot \mu\nabla u \cdot \boldsymbol{n}\, d\Gamma + \int_\Omega \nabla\delta u \cdot (\lambda + \mu)(\nabla \cdot u)\, d\Omega - \int_{\partial\Omega} \delta u \cdot (\lambda + \mu)(\nabla \cdot u) \cdot \boldsymbol{n}\, d\Gamma$$
$$= \int_\Omega \delta u \cdot \rho b\, d\Omega$$

Again, $u, \delta u \in H^1(\Omega)$ due to the first term

$$\int_\Omega \nabla\delta u \cdot \mu\nabla u\, d\Omega$$

- **3rd formulation**

$$\int_\Omega \delta u \cdot \big(\mu\nabla \times (\nabla \times u) - (\lambda + 2\mu)\nabla(\nabla \cdot u)\big) d\Omega = \int_\Omega \delta u \cdot \rho b\, d\Omega$$

The left hand side is integrated by parts using the same identity than for the Maxwell problem:

$$\int_{\Omega} \delta u \cdot \big(\mu \nabla \times (\nabla \times u) - (\lambda + 2\mu) \nabla (\nabla \cdot u)\big) d\Omega$$
$$= \int_{\Omega} \nabla \times \delta u \cdot \mu (\nabla \times u) d\Omega - \int_{\partial\Omega} \delta u \cdot \mu (\nabla \times u) \cdot \boldsymbol{n} d\Gamma + \int_{\Omega} \nabla \delta u \cdot (\lambda + 2\mu)(\nabla \cdot u)\, d\Omega$$
$$- \int_{\partial\Omega} \delta u \cdot (\lambda + 2\mu)(\nabla \cdot u) \cdot \boldsymbol{n}\, d\Gamma$$

This result in the following weak form:

$$\int_{\Omega} \nabla \times \delta u \cdot \mu (\nabla \times u) d\Omega - \int_{\partial\Omega} \delta u \cdot \mu (\nabla \times u) \cdot \boldsymbol{n} d\Gamma + \int_{\Omega} \nabla \delta u \cdot (\lambda + 2\mu)(\nabla \cdot u)\, d\Omega$$
$$- \int_{\partial\Omega} \delta u \cdot (\lambda + 2\mu)(\nabla \cdot u) \cdot \boldsymbol{n}\, d\Gamma = \int_{\Omega} \delta u \cdot \rho b\, d\Omega$$

The third term contains a gradient of δu for this reason $\delta u \in H^1(\Omega)$. In the case of u there are terms with the divergence of u and other with the curl of u. For this reason: $u \in H^1(curl, \Omega) \cap H^1(div, \Omega)$. $u \in H^1(curl, \Omega)$ implies $u_{i,j} \in H^1(\Omega)\ for\ i \neq j. u \in H^1(div, \Omega)$ implies $u_{i,j} \in H^1(\Omega)\ for\ i = j$. For this reason, : $u \in H^1(curl, \Omega) \cap H^1(div, \Omega) = H^1(\Omega)$.

In all cases, as all boundary is of Dirichlet type, the surface integrals vanish.

b) If $\boldsymbol{\Gamma}$ is a surface that intersects $\boldsymbol{\Omega}$, obtain the transmission conditions across $\boldsymbol{\Gamma}$ that follow by imposing in the variational form of the problem that the integral is additive.

- **1st formulation**

$$\int_{\Omega_1} \delta u \cdot \Big(-2\mu \nabla \cdot \big(\varepsilon(u)\big) - \lambda \nabla (\nabla \cdot u)\Big)\, d\Omega + \int_{\Omega_2} \delta u \cdot \Big(-2\mu \nabla \cdot \big(\varepsilon(u)\big) - \lambda \nabla (\nabla \cdot u)\Big)\, d\Omega$$
$$= \int_{\Omega} \delta u \cdot \rho b\, d\Omega$$

Integrating by parts:

$$\int_{\Omega_1} \delta u \cdot \Big(-2\mu \nabla \cdot \big(\varepsilon(u)\big) - \lambda \nabla (\nabla \cdot u)\Big)\, d\Omega + \int_{\Omega_2} \delta u \cdot \Big(-2\mu \nabla \cdot \big(\varepsilon(u)\big) - \lambda \nabla (\nabla \cdot u)\Big)\, d\Omega$$
$$= \int_{\Omega_1} \nabla \delta u : 2\mu\big(\varepsilon(u)\big)\, d\Omega - \int_{\Gamma} \delta u \cdot\ 2\mu\varepsilon(u)|_{\Omega_1} \cdot \boldsymbol{n}\, d\Gamma + \int_{\Omega_1} \nabla \delta u \cdot\ \lambda(\nabla \cdot u)\, d\Omega$$
$$- \int_{\Gamma} \delta u \cdot\ \lambda\big(\nabla \cdot u|_{\Omega_1}\big) \cdot \boldsymbol{n}\, d\Gamma + \int_{\Omega_2} \nabla \delta u : 2\mu\big(\varepsilon(u)\big)\, d\Omega - \int_{\Gamma} \delta u \cdot\ 2\mu\varepsilon(u)|_{\Omega_2} \cdot \boldsymbol{n}\, d\Gamma$$
$$+ \int_{\Omega_2} \nabla \delta u \cdot\ \lambda(\nabla \cdot u)\, d\Omega - \int_{\Gamma} \delta u \cdot\ \lambda\big(\nabla \cdot u|_{\Omega_2}\big) \cdot \boldsymbol{n}\, d\Gamma$$

The transmission conditions are:

$$[\![\mu\big(\varepsilon(u)\big) \cdot \boldsymbol{n}]\!] = 0$$
$$[\![\lambda(\nabla \cdot u) \cdot \boldsymbol{n}]\!] = 0$$

- **2nd formulation**

$$\int_{\Omega_1} \delta u \cdot \big(-\mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u)\big) d\Omega + \int_{\Omega_2} \delta u \cdot \big(-\mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u)\big) d\Omega = \int_{\Omega} \delta u \cdot \rho b \, d\Omega$$

Integrating by parts:

$$\begin{aligned}
&\int_{\Omega_1} \delta u \cdot \big(-\mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u)\big) d\Omega + \int_{\Omega_2} \delta u \cdot \big(-\mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u)\big) d\Omega \\
&\qquad = \int_{\Omega_1} \nabla\delta u \cdot \mu\nabla u \, d\Omega - \int_{\Gamma} \delta u \cdot \mu\nabla u|_{\Omega_1} \cdot \boldsymbol{n} d\Gamma + \int_{\Omega_1} \nabla\delta u \cdot (\lambda+\mu)(\nabla\cdot u) \, d\Omega \\
&\qquad - \int_{\Gamma} \delta u \cdot (\lambda+\mu)\big(\nabla\cdot u|_{\Omega_1}\big) \cdot \boldsymbol{n} \, d\Gamma + \int_{\Omega_2} \nabla\delta u \cdot \mu\nabla u \, d\Omega - \int_{\Gamma} \delta u \cdot \mu\nabla u|_{\Omega_2} \cdot \boldsymbol{n} d\Gamma \\
&\qquad + \int_{\Omega_2} \nabla\delta u \cdot (\lambda+\mu)(\nabla\cdot u) \, d\Omega - \int_{\Gamma} \delta u \cdot (\lambda+\mu)\big(\nabla\cdot u|_{\Omega_2}\big) \cdot \boldsymbol{n} \, d\Gamma
\end{aligned}$$

Resulting in the following transmission conditions:

$$[\![\mu\nabla u \cdot \boldsymbol{n}]\!] = 0$$

$$[\![(\lambda+\mu)(\nabla\cdot u) \cdot \boldsymbol{n}]\!] = 0$$

- **3rd formulation**

$$\begin{aligned}
&\int_{\Omega_1} \delta u \cdot \big(\mu\nabla\times(\nabla\times u) - (\lambda+2\mu)\nabla(\nabla\cdot u)\big) d\Omega + \int_{\Omega_2} \delta u \cdot \big(\mu\nabla\times(\nabla\times u) - (\lambda+2\mu)\nabla(\nabla\cdot u)\big) d\Omega \\
&\qquad = \int_{\Omega} \delta u \cdot \rho b \, d\Omega
\end{aligned}$$

Integrating by parts:

$$\begin{aligned}
&\int_{\Omega_1} \delta u \cdot \big(\mu\nabla\times(\nabla\times u) - (\lambda+2\mu)\nabla(\nabla\cdot u)\big) d\Omega + \int_{\Omega_2} \delta u \cdot \big(\mu\nabla\times(\nabla\times u) - (\lambda+2\mu)\nabla(\nabla\cdot u)\big) d\Omega \\
&\qquad = \int_{\Omega_1} \nabla\times\delta u \cdot \mu(\nabla\times u) d\Omega - \int_{\partial\Omega} \delta u \cdot \mu\big(\nabla\times u|_{\Omega_1}\big) \cdot \boldsymbol{n} d\Gamma \\
&\qquad + \int_{\Omega_1} \nabla\delta u \cdot (\lambda+2\mu)(\nabla\cdot u) \, d\Omega - \int_{\partial\Omega} \delta u \cdot (\lambda+2\mu)\big(\nabla\cdot u|_{\Omega_1}\big) \cdot \boldsymbol{n} \, d\Gamma \\
&\qquad + \int_{\Omega_2} \nabla\times\delta u \cdot \mu(\nabla\times u) d\Omega - \int_{\partial\Omega} \delta u \cdot \mu\big(\nabla\times u|_{\Omega_2}\big) \cdot \boldsymbol{n} d\Gamma \\
&\qquad + \int_{\Omega_2} \nabla\delta u \cdot (\lambda+2\mu)(\nabla\cdot u) \, d\Omega - \int_{\partial\Omega} \delta u \cdot (\lambda+2\mu)\big(\nabla\cdot u|_{\Omega_2}\big) \cdot \boldsymbol{n} \, d\Gamma
\end{aligned}$$

The transmission conditions are:

$$[\![\mu(\nabla\times u) \cdot \boldsymbol{n}]\!] = 0$$

$$[\![(\lambda+2\mu)(\nabla\cdot u) \cdot \boldsymbol{n}]\!] = 0$$

2. Transmission conditions

2.1. Consider Problem 1 of Section 1. Let $[0, L] = [0, L_1] \cup [L_2, L]$, with $L_2 < L_1$.

a) Write down an iteration-by-subdomain scheme based on a Schwarz additive domain decomposition method.

Each iteration is divided in two steps, one per subdomain:

- **1st subdomain**

$$EI \frac{d^4 v_1^k}{dx^4} = f \; in \; [0, L_1]$$

$$v_1^k(0) = \bar{v}_0$$

$$\frac{dv_1^k}{dx}(0) = \bar{v}_{x0}$$

$$v_1^k(L_1) = v_2^{(k-1)}(L_1)$$

$$\frac{dv_1^k}{dx}(L_1) = \frac{dv_2^{(k-1)}}{dx}(L_1)$$

- **2nd subdomain**

$$EI \frac{d^4 v_2^k}{dx^4} = f \; in \; [L_2, L]$$

$$v_2^k(L) = \bar{v}_L$$

$$\frac{dv_2^k}{dx}(L) = \bar{v}_{xL}$$

$$v_2^k(L_2) = v_1^l(L_2)$$

$$\frac{dv_2^k}{dx}(L_2) = \frac{dv_1^l}{dx}(L_2)$$

Here, $l = k - 1$ for a Jacobi scheme and $l = k$ for a Gauss-Seidel scheme. After each iteration:

$$v(x) = \begin{cases} v_1(x) \; for \; 0 \leq x \leq L_2 \\ v_2(x) \; for L_2 \leq x \leq L \end{cases}$$

b) Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

Here the Jacobi scheme will be explained. The two steps are analysed separately:

- **1st subdomain**

$$A_1 v = f_1$$

Here, the contributions of the Dirichlet boundary conditions at the interface are added at the force vector. The matrix A_1 and vector f_1 are constructed as

$$[A_1]_{ij} = a_1(N_i, N_j)$$

$$[f_1]_i = [f_1^I]_i + v_2^{(k-1)}(L_1) \cdot [f_1^\Gamma]_i = l_1(N_i) - v_2^{(k-1)}(L_1) \cdot a_1(N_i, N_1^\Gamma)$$

$$a_1(v,u) = \int_0^{L_1} EI \frac{d^2v}{dx^2}\frac{d^2u}{dx^2}\, dx$$

$$l_1(v) = \int_0^{L_1} fu\, dx$$

- **2nd subdomain**

$$A_2 v_2 = f_2$$

Where

$$[A_2]_{ij} = a_2(N_i, N_j)$$

$$[f_2]_i = [f_2^I]_i + v_1^{(k-1)}(L_2) \cdot [f_2^\Gamma]_i = l_2(N_i) - a_2(N_i, N_2^\Gamma)$$

$$a_1(v,u) = \int_{L_2}^{L} EI \frac{d^2v}{dx^2}\frac{d^2u}{dx^2}\, dx$$

$$l_1(v) = \int_{L_2}^{L} fu\, dx$$

The values at the interface can be obtained with the corresponding reduction operator:

$$v(L_2) = R_2^\Gamma u$$

$$v(L_1) = R_1^\Gamma u$$

The solution after each iteration computed with two injection operators applied on the two subdomain solutions:

$$\begin{aligned} v^{k+1} = I_1 v_1^{k+1} + I_2 v_2^{k+1} &= I_1(A_1^{-1}f_1) + I_2(A_2^{-1}f_2) = I_1A_1^{-1}(f_1^I + f_1^\Gamma R_2^\Gamma v^k) + I_2A_2^{-1}(f_2^I + f_2^\Gamma R_1^\Gamma v^k) \\ &= (I_1A_1^{-1}f_1^I + I_2A_2^{-1}f_2^I) + (I_1A_1^{-1}f_1^\Gamma R_2^\Gamma + I_2A_2^{-1}f_2^\Gamma R_1^\Gamma)v^k \end{aligned}$$

Here it is seen two different terms, one constant representing the body force and a second one depending on v^k that is the one that ensures the transmission conditions.

2.2. Consider Problem 2 of Section 1. Let Γ be a surface that intersects Ω.

a) Write down an iteration-by-subdomain scheme based on the Dirichlet-Neumann coupling.

The domain of the problem is divided in two non-intersecting subdomains:

$$\Omega_1 \cup \Omega_2 = \Omega$$

$$\Omega_1 \cap \Omega_2 = \Gamma_{12}$$

$$\Gamma_i = \partial\Omega \cap \Omega_i$$

The subdomain 1 is solved as a Neumann problem where the transmission condition of fluxes is enforced:

$$\nu\nabla\times\nabla\times\boldsymbol{u}_1^{\boldsymbol{k}} = \boldsymbol{f}\ in\ \Omega_1$$
$$\boldsymbol{n}\times\boldsymbol{u}_1^k = \boldsymbol{0}\ on\ \Gamma_1$$
$$\nu_1(\nabla\times\boldsymbol{u}_1^k)\cdot\boldsymbol{n} = \nu_2\left(\nabla\times\boldsymbol{u}_2^{(k-1)}\right)\cdot\boldsymbol{n}\ on\ \Gamma_{12}$$

Subdomain 2 is solved as a Dirichlet problem enforcing the transmission condition of the primal variable:

$$\nu\nabla\times\nabla\times\boldsymbol{u}_2^{\boldsymbol{k}} = \boldsymbol{f}\ in\ \Omega_2$$
$$\boldsymbol{n}\times\boldsymbol{u}_2^k = \boldsymbol{0}\ on\ \Gamma_2$$
$$\boldsymbol{n}\times\boldsymbol{u}_2^k = \boldsymbol{n}\times\boldsymbol{u}_1^l\ on\ \Gamma_{12}$$

$l = k-1$ for a Jacobi scheme and $l = k$ for a Gauss-Seidel scheme.

Note than the equation enforcing u is divergence free has been omitted as it is redundant.

b) Obtain the expression of the Steklov-Poincaré operator of the problem.

As the problem is linear the solution can be expressed as the sum of a problem where the solution vanishes at the boundary and a homogeneous problem with the correct boundary conditions. Splitting the solution at each subdomain:

$$\boldsymbol{u}_i = \boldsymbol{u}_i^0 + \widetilde{\boldsymbol{u}_\iota}\ for\ i = 1,2$$

$$\nu\nabla\times\nabla\times\boldsymbol{u}_i^{\boldsymbol{0}} = \boldsymbol{f}\ in\ \Omega_{\mathrm{i}}$$
$$\boldsymbol{n}\times\boldsymbol{u}_i^0 = \boldsymbol{0}\ on\ \Gamma_i$$
$$\boldsymbol{n}\times\boldsymbol{u}_i^0 = \boldsymbol{0}\ on\ \Gamma_{12}$$

$$\nu\nabla\times\nabla\times\widetilde{\boldsymbol{u}_\iota} = \boldsymbol{0}\ in\ \Omega_{\mathrm{i}}$$
$$\boldsymbol{n}\times\widetilde{\boldsymbol{u}_\iota} = \boldsymbol{0}\ on\ \Gamma_i$$
$$\boldsymbol{n}\times\widetilde{\boldsymbol{u}_\iota} = \boldsymbol{\varphi}\ on\ \Gamma_{12}$$

With this definition of the problem it is ensured the transmission condition of the primal variable $[\![\boldsymbol{n}\times\boldsymbol{u}]\!] = \boldsymbol{0}$.

The transmission conditions for fluxes is $[\![\nu(\boldsymbol{\nabla}\times\boldsymbol{u})\cdot\boldsymbol{n}]\!] = \boldsymbol{0}$. So the problem is to find the trace of $\boldsymbol{u}$ on Γ_{12} such that the transmission condition is ensured:

$$\nu_1(\boldsymbol{\nabla}\times\boldsymbol{u_1})\cdot\boldsymbol{n} = \nu_2(\boldsymbol{\nabla}\times\boldsymbol{u_2})\cdot\boldsymbol{n} \rightarrow \nu_1(\boldsymbol{\nabla}\times\widetilde{\boldsymbol{u}}_{\boldsymbol{1}})\cdot\boldsymbol{n} - \nu_2(\boldsymbol{\nabla}\times\widetilde{\boldsymbol{u}}_{\boldsymbol{2}})\cdot\boldsymbol{n}$$
$$= -\nu_1(\boldsymbol{\nabla}\times\boldsymbol{u}_{\boldsymbol{1}}^{\boldsymbol{0}})\cdot\boldsymbol{n} + \nu_2(\boldsymbol{\nabla}\times\boldsymbol{u}_{\boldsymbol{2}}^{\boldsymbol{0}})\cdot\boldsymbol{n}$$

The Steklov-Poincaré operator is defined as

$$S: H^{\frac{1}{2}}(curl,\Gamma_{12}) \rightarrow H^{-\frac{1}{2}}(curl,\Gamma_{12})$$

$$\varphi \rightarrow \nu_1(\boldsymbol{\nabla}\times\widetilde{\boldsymbol{u}}_{\boldsymbol{1}})\cdot\boldsymbol{n} - \nu_2(\boldsymbol{\nabla}\times\widetilde{\boldsymbol{u}}_{\boldsymbol{2}})\cdot\boldsymbol{n}$$

And

$$G = -\nu_1(\boldsymbol{\nabla}\times\boldsymbol{u}_{\boldsymbol{1}}^{\boldsymbol{0}})\cdot\boldsymbol{n} + \nu_2(\boldsymbol{\nabla}\times\boldsymbol{u}_{\boldsymbol{2}}^{\boldsymbol{0}})\cdot\boldsymbol{n}$$

In order to ensure the transmission conditions, the following problem must be solved:

$$S\varphi = G$$

c) **Obtain the matrix version of the previous scheme once space has been discretized using finite elements.**

The Neumann problem for the first subdomain can be written as:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} \end{bmatrix}\begin{bmatrix} U_1 \\ U_\Gamma \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2}U_2 \end{bmatrix}$$

The Dirichlet subdomain problem is written as:

$$A_{22}U_2 = F_2 - A_{2\Gamma}U_\Gamma$$

The three equations can be written in a single system:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} & 0 \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ 0 & A_{2\Gamma} & A_{22} \end{bmatrix}\begin{bmatrix} U_1 \\ U_\Gamma \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma \\ F_2 \end{bmatrix}$$

Using the iterative scheme:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix}\begin{bmatrix} U_1^k \\ U_\Gamma^k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2}U_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)}U_\Gamma^{(k-1)} \end{bmatrix}$$

The Dirichlet problem is written using a Jacobi or Gauss-Seidel scheme defining l as explained before:

$$A_{22}U_2^k = F_2 - A_{2\Gamma}U_\Gamma^l$$

2.3. Consider the problem of finding $u: \Omega \to \mathbb{R}$ such that

$$-k\Delta u = f \ in \ \Omega$$

$$u = 0 \ on \ \partial\Omega$$

Where $k > 0$. Let Γ be a surface crossing Ω.

a) **Write down an iteration-by-subdomain scheme based on the Dirichlet-Robin coupling.**

The same notation as in the previous problem is used for $\Omega_1, \Omega_2, \Gamma_1, \Gamma_2$ and Γ_{12}.

The Dirichlet problem is written as

$$-k\Delta u_1^k = f \ in \ \Omega_1$$

$$u_1^k = 0 \ on \ \Gamma_1$$

$$u_1^k = u_2^{(k-1)} \ on \ \Gamma_{12}$$

The Robin problem is

$$-k\Delta u_2^k = f \ in \ \Omega_2$$

$$u_2^k = 0 \ on \ \Gamma_2$$

$$k_2\left(\nabla u_2^k \cdot \boldsymbol{n}\right) + \gamma u_2^k = k_2\left(\nabla u_1^l \cdot \boldsymbol{n}\right) + \gamma u_1^l \ on \ \Gamma_{12}$$

Here, $\gamma > 0$ and $l = k - 1$ for a Jacobi scheme and $l = k$ for Gauss-Seidel scheme.

b) Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

The Dirichlet problems is written as:

$$A_{22}U_2 = F_2 - A_{2\Gamma}U_\Gamma$$

The Robin problem is very similar to the Neumann one:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} + \gamma I \end{bmatrix} \begin{bmatrix} U_1 \\ U_\Gamma \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 2}U_2 - \left(A_{\Gamma\Gamma}^{(2)} - \gamma I\right) U_\Gamma \end{bmatrix}$$

The difference is that the submatrix $A_{\Gamma 1}$ accounts for the additional terms of the Robin BC.

Using the iterative scheme:

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} + \gamma I \end{bmatrix} \begin{bmatrix} U_1^k \\ U_\Gamma^k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_\Gamma - A_{\Gamma 1}U_2^{(k-1)} - \left(A_{\Gamma\Gamma}^{(2)} - \gamma I\right) U_\Gamma^{(k-1)} \end{bmatrix}$$

$$A_{22}U_2^k = F_2 - A_{2\Gamma}U_\Gamma^l$$

c) Obtain the Schur complement as discrete version of the Steklov-Poicaré operator.

First local problems are solved taking into account the force term:

$$-k\Delta u_i^0 = f \ in \ \Omega_i$$

$$u_i^0 = 0 \ on \ \Gamma_i$$

$$u_i^0 = 0 \ on \ \Gamma_{12}$$

Then G is computed as:

$$G = -k_1(\nabla u_1^0 \cdot \boldsymbol{n}) + k_2(\nabla u_2^0 \cdot \boldsymbol{n})$$

The discrete version of this problems are:

$$A_{11}U_1^0 = F_1$$

$$A_{22}U_2^0 = F_2$$

The Steklov-Poincaré operator is, given the trace of u on Γ_{12} (φ), solve

$$-k\Delta\tilde{u}_i = 0 \; in \; \Omega_i$$

$$\tilde{u}_i = 0 \; on \; \Gamma_i$$

$$\tilde{u}_i = \varphi \; on \; \Gamma_{12}$$

The discrete version of this problem is:

$$A_{11}\widetilde{U}_1 + A_{1\Gamma}U_\Gamma = 0$$

$$A_{22}\widetilde{U}_2 + A_{2\Gamma}U_\Gamma = 0$$

Where φ is discretised as U_Γ.

Then, $S\varphi$ is computed as

$$S\varphi = k_1(\nabla\tilde{u}_1 \cdot \boldsymbol{n}) - k_2(\nabla\tilde{u}_2 \cdot \boldsymbol{n})$$

The discrete version of G is:

$$G = F_\Gamma - A_{\Gamma 1}U_1^0 - A_{\Gamma 2}U_2^0$$

Where the minus sign is to note that it has the reverse sign of $S\varphi$. Substituting U_1^0 and U_2^0 using the linear problem computed before:

$$G = F_\Gamma - A_{\Gamma 1}A_{11}^{-1}F_1 - A_{\Gamma 2}A_{22}^{-1}F_2$$

The discrete version of $S\varphi$ is:

$$S\varphi = A_{\Gamma\Gamma}U_\Gamma + A_{\Gamma 1}\widetilde{U}_1 + A_{\Gamma 2}\widetilde{U}_2$$

Substituting:

$$S\varphi = (A_{\Gamma\Gamma} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma})U_\Gamma$$

Another point to note is that in the Steklov-Poincaré problem φ has a measure of 0 and does not contribute to the problem. However in the discrete version U_Γ have contributions to the problem. Because of that $A_{\Gamma\Gamma}$ and F_Γ are added.

The continuous problem is:

$$S\varphi = G$$

The discrete version is:

$$(A_{\Gamma\Gamma} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma})U_\Gamma = F_\Gamma - A_{\Gamma 1}U_1^0 - A_{\Gamma 2}U_2^0$$

d) Identify the preconditioner for the Schur complement equation arising from the iterative scheme of section (a).

First, let divide the Schur complement as

$$S = S_1 + S_2$$

$$S_1 = A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma}$$

$$S_2 = A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma}$$

We will consider the Gauss-Seidel scheme:

$$U_1^k = A_{11}^{-1}\left(F_1 - A_{1\Gamma}U_\Gamma^k\right)$$

$$U_2^k = A_{22}^{-1}\left(F_2 - A_{2\Gamma}U_\Gamma^k\right)$$

$$A_{\Gamma 1}A_{11}^{-1}\left(F_1 - A_{1\Gamma}U_\Gamma^k\right) + A_{\Gamma\Gamma}^{(1)}U_\Gamma^k + \gamma U_\Gamma^k = F_\Gamma - A_{\Gamma 1}A_{22}^{-1}\left(F_2 - A_{2\Gamma}U_\Gamma^{(k-1)}\right) - \left(A_{\Gamma\Gamma}^{(2)} - \gamma I\right)U_\Gamma^{(k-1)}$$

$$\begin{aligned}(S_1 + \gamma I)U_\Gamma^k = (F_\Gamma - A_{\Gamma 1}A_{11}^{-1}F_1 - A_{\Gamma 1}A_{22}^{-1}F_2) - \left(A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 1}A_{22}^{-1}A_{2\Gamma} + A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma}\right)U_\Gamma^{(k-1)} \\ + \left(A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma}\right)U_\Gamma^{(k-1)} + \gamma(A_{22}^{-1}U_\Gamma^{(k-1)}\end{aligned}$$

$$(S_1 + \gamma I)U_\Gamma^k = G - SU_\Gamma^{(k-1)} + (S_1 + \gamma I)U_\Gamma^{(k-1)}$$

$$U_\Gamma^k = U_\Gamma^{(k-1)} + (S_1 + \gamma I)^{-1}\left(G - SU_\Gamma^{(k-1)}\right)$$

So the preconditioner of the Dirichlet-Robinson algorithm is

$$P_{DR} = (S_1 + \gamma I)^{-1}$$

3. Coupling of heterogeneous problems

3.1. Consider the beam described in Problem 1 of Section 1. Apart from being clamped at $x = 0$ and $x = L$, the beam is supported on an elastic wall that occupies the square $[0, L] \times [-L, 0]$, where $y = 0$ corresponds to the beam axis. The wall is clamped everywhere except on the upper wall, where the beam is. The wall displacements in the *x*- and *y*- directions are u and v, respectively, and the elastic properties E (Young modulus) and ν (Poisson's coefficient). No loads are applied on the wall, except for those coming from the beam.

a) Write down the equations in the wall assuming a plane stress behaviour.

As no body forces are applied on the wall, the equation of equilibrium is:

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \; in \; \Omega_{Wall}$$

The stress can be written in terms of the stress with the constitutive matrix:

$$\boldsymbol{\sigma} = \boldsymbol{C\varepsilon}$$

Where

$$\boldsymbol{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}$$

And the strain is the symmetric gradient of the displacement:

$$\varepsilon = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$$

With this, the PDE governing the behaviour of the wall displacement is:

$$\nabla \cdot \left(\frac{C}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)\right) = 0 \; in \; \Omega_{Wall}$$

The BCs are that the displacements are 0 on the clamped sides, which will be denoted as Γ_D:

$$\boldsymbol{u} = 0 \; on \; \Gamma_D$$

At the upper side, the transmission conditions have to be enforced.

b) Write down the equations for the beam modified because of the presence of the wall.

The differential equation to solve is the same than before but with a contribution to the force term from the wall:

$$EI\frac{d^4 v}{dx^4} = f + t_{wall}^y \; in \; \Omega_{Beam}$$

Where $t \ll L$ is the thickness of the wall.

The changes BCs of the wall remain the same:

$$v_{Beam}(0,0) = 0$$

$$\frac{dv_{Beam}}{dx}(0,0) = 0$$

$$v_{Beam}(L,0) = 0$$

$$\frac{dv_{Beam}}{dx}(L,0) = 0$$

$$u_{Beam}(0,0) = u_{Wall}(0,0)$$

c) **Obtain the adequate transmission conditions for v and the normal component of the traction on the wall at** $y = 0$.

In this problem, the equations of the beam cannot be obtained as the limit case of the plane stress problem with an infinitesimal height. For that reason, transmission conditions are obtained from physical principles. The first transmission condition is that the displacements must be continuous at $(x, 0)$:

$$[\![v]\!] = 0$$

The normal component of the traction from the beam is computed as $t^x_{Beam} = EA\frac{du_{Wall}}{dx}$.

The normal component per unit length is computed as $t^y_{Wall}(x) = [0,1] \cdot (\boldsymbol{\sigma}(x,0) \cdot \boldsymbol{n})$. The transmission condition is to enforce continuity of tractions:

$$[\![t^y \cdot \boldsymbol{n}]\!] = 0$$

To impose it, the term t^y_{Wall} of the beam equation has to be equal than $t^y_{Wall}(x) = [0,1] \cdot (\boldsymbol{\sigma}(x,0) \cdot \boldsymbol{n})$**.**

d) ***Suggest*** **transmission conditions for u and the tangent component of the traction on the wall at** $y = 0$. **Discuss the implications if this component is not assumed to be zero.**

The transmission condition for u is the same as for v: continuity must be enforced.

About the transmission condition for the tangential component of the traction can be enforced in a similar way than the normal. The problem is that there is not differential equation about the tangential traction. The problem to find the displacements in the *x*- direction such that.

$$EA\frac{d^2u}{dx^2} = f_x \ in \ \Omega_{Beam}$$

The BCs are:

$$u_{Beam}(0,0) = 0$$

$$u_{Beam}(L,0) = 0$$

With this, the traction in tangential component can be enforced.

3.2. Let S_D and S_S be the Dirichlet-to-Neumann operators for the Darcy and the Stokes problems, respectively (see the class notes, chapter 3). The Steklov-Poincaré equation can be written as:

$$S_S(\lambda) = S_D(\lambda),$$

Where λ is the normal velocity on Γ, the interface between the Darcy and the Stokes regions.

a) Obtain the discrete version of the previous equation when space is discretized using finite elements. Relate the resulting matrices to those arising from the discretization of the Darcy and the Stokes problems separately.

Simple Dirichlet BCs will be imposed.

The strong form of the Stokes problem is:

$$-\nu\Delta u_s + \nabla p = f \ \ in\ \Omega_s$$

$$\nabla \cdot u_s = 0 \ \ in\ \Omega_s$$

$$u_s = 0 \ \ on\ \Gamma_s$$

The strong form of the Darcy problem is:

$$u_D + \kappa\nabla\varphi = 0 \ \ in\ \Omega_D$$

$$\nabla \cdot u_D = 0 \ \ in\ \Omega_D$$

$$u_D = 0\ on\ \Gamma_D$$

Apart of that, the transmission conditions have to be imposed as

$$S_S(\lambda) = S_D(\lambda)$$

The discretization of the Stokes problem is:

$$\int_{\Omega_s} \delta u_s \cdot (-\nu\Delta u_s + \nabla p)\, d\Omega = \int_{\Omega_s} \delta u_s \cdot f\, d\Omega$$

$$\int_{\Omega_s} \delta p \cdot (\nabla \cdot u_s)\, d\Omega = 0$$

Integrating the first equation by parts:

$$\int_{\Omega_s} \delta u_s \cdot (-\nu\Delta u_s + \nabla p)\, d\Omega$$
$$= \int_{\Omega_s} \nabla\delta u_s \cdot \nu\nabla u_s\, d\Omega - \int_{\Omega_s} (\nabla \cdot \delta u_s) \cdot p\, d\Omega - \int_{\partial\Omega_s} \delta u_s \cdot (\nu\nabla u_s - pI) \cdot n\, d\Gamma$$

This results in:

$$\int_{\Omega_s} \nabla\delta u_s \cdot \nu\nabla u_s\, d\Omega - \int_{\Omega_s} (\nabla \cdot \delta u_s) \cdot p\, d\Omega = \int_{\Omega_s} \delta u_s \cdot f\, d\Omega + \int_{\partial\Omega_s} \delta u_s \cdot (\nu\nabla u_s - pI) \cdot n\, d\Gamma$$

The resulting linear system is:

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{G_s} \\ \boldsymbol{G_s^T} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u_s} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_s} \\ \boldsymbol{0} \end{bmatrix}$$

The space of functions is $u_s \in H^1(\Omega_s)^d, p \in L_2(\Omega_s)$

The discretization of the Darcy problem is:

$$\int_{\Omega_D} \delta u_D \cdot (\kappa^{-1} u_D + \nabla\varphi)\, d\Omega = 0$$

$$\int_{\Omega_D} \delta\varphi \cdot (\nabla \cdot u_D)\, d\Omega = 0$$

Integrating by parts the first equation:

$$\int_{\Omega_D} \delta u_D \cdot (\kappa^{-1} u_D + \nabla\varphi)\, d\Omega = \int_{\Omega_D} \delta u_D \cdot \kappa^{-1} u_D\, d\Omega - \int_{\Omega_D} (\nabla \cdot \delta u_D) \cdot \varphi\, d\Omega + \int_{\partial\Omega_D} \delta u_D \cdot \varphi \cdot n\, d\Gamma$$

The resulting system of equations is:

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{G_D} \\ \boldsymbol{G_D^T} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u_S} \\ \boldsymbol{\varphi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_D} \\ \boldsymbol{0} \end{bmatrix}$$

The Steklov-Poincaré equation can be discretized as well:

$$S_S(\lambda) - S_D(\lambda) = 0$$

$$\begin{bmatrix} \boldsymbol{A_{SS}} & \boldsymbol{A_{SD}} \\ \boldsymbol{A_{DS}} & \boldsymbol{A_{DD}} \end{bmatrix} \begin{bmatrix} \boldsymbol{u_s^\Gamma} \\ \boldsymbol{u_D^\Gamma} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_s} \\ \boldsymbol{f_D} \end{bmatrix}$$

b) **Write down the matrix of a Dirichlet-Neumann iteration-by-subdomain using the matrices of the Darcy and the Stokes problems**.

First, given a guess for the velocity at the interface, the Stokes equation is solved using Neumann BC:

$$\begin{bmatrix} \boldsymbol{K_{ss}} & \boldsymbol{K_{s\Gamma}} & \boldsymbol{G_s} \\ \boldsymbol{K_{\Gamma s}} & \boldsymbol{K_{\Gamma\Gamma}} & \boldsymbol{G_{\Gamma p}} \\ \boldsymbol{G_s^T} & \boldsymbol{G_{\Gamma p}^T} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u_s^k} \\ \boldsymbol{u_\Gamma^k} \\ \boldsymbol{p^k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_s} \\ \boldsymbol{f_\Gamma} - \boldsymbol{M_{\Gamma D} u_D^{(k-1)}} - \boldsymbol{G_{\Gamma\varphi}\varphi^{(k-1)}} - \boldsymbol{M_{\Gamma\Gamma} u_\Gamma^{(k-1)}} \\ \boldsymbol{0} \end{bmatrix}$$

Now, the Darcy problem is solved using Dirichlet BC:

$$\begin{bmatrix} \boldsymbol{M_{DD}} & \boldsymbol{G_D} \\ \boldsymbol{G_D^T} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u_D^k} \\ \boldsymbol{\varphi^k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_D} - \boldsymbol{A_{D\Gamma} u_\Gamma^k} \\ \boldsymbol{0} \end{bmatrix}$$

A Gauss-Seidel scheme has been used.

c) Identify the Richardson iteration for the algebraic problem in (a) resulting from (b).

From the Neumann equation, $\boldsymbol{u_s^k}$ and $\boldsymbol{p}$ can be taken as function of $\boldsymbol{u_\Gamma^k}$:

$$\begin{bmatrix} \boldsymbol{K_{ss}} & \boldsymbol{G_s} \\ \boldsymbol{G_s^T} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u_s^k} \\ \boldsymbol{p^k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_s} - \boldsymbol{K_{s\Gamma} u_\Gamma^k} \\ -\boldsymbol{G_{\Gamma p}^T u_\Gamma^k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_s} \\ \boldsymbol{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{K_{s\Gamma}} \\ \boldsymbol{G_{\Gamma p}^T} \end{bmatrix} \boldsymbol{u_\Gamma^k}$$

The Dirichlet problem is already written as function of $\boldsymbol{u_\Gamma^k}$

$$\begin{bmatrix} \boldsymbol{M_{DD}} & \boldsymbol{G_D} \\ \boldsymbol{G_D^T} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u_D^k} \\ \boldsymbol{\varphi^k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_D} - \boldsymbol{A_{D\Gamma} u_\Gamma^k} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f_D} \\ \boldsymbol{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{A_{D\Gamma}} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{u_\Gamma^k}$$

From the Neumann equation, $\boldsymbol{u}_\Gamma^k$ is computed as:

$$\boldsymbol{K}_{\Gamma s}\boldsymbol{u}_s^k + \boldsymbol{K}_{\Gamma\Gamma}\boldsymbol{u}_\Gamma^k + \boldsymbol{G}_{\Gamma p}\boldsymbol{p}^k = \boldsymbol{f}_\Gamma - \boldsymbol{M}_{\Gamma D}\boldsymbol{u}_D^{(k-1)} - \boldsymbol{G}_{\Gamma\varphi}\boldsymbol{\varphi}^{(k-1)} - \boldsymbol{M}_{\Gamma\Gamma}\boldsymbol{u}_\Gamma^{(k-1)}$$

$$[\boldsymbol{K}_{\Gamma s} \quad \boldsymbol{G}_{\Gamma p}]\begin{bmatrix}\boldsymbol{u}_s^k\\ \boldsymbol{p}^k\end{bmatrix} + \boldsymbol{K}_{\Gamma\Gamma}\boldsymbol{u}_\Gamma^k = \boldsymbol{f}_\Gamma - [\boldsymbol{M}_{\Gamma D} \quad \boldsymbol{G}_{\Gamma\varphi}]\begin{bmatrix}\boldsymbol{u}_D^{(k-1)}\\ \boldsymbol{\varphi}^{(k-1)}\end{bmatrix} - \boldsymbol{M}_{\Gamma\Gamma}\boldsymbol{u}_\Gamma^{(k-1)}$$

$$\begin{aligned}
&[\boldsymbol{K}_{\Gamma s} \quad \boldsymbol{G}_{\Gamma p}]\begin{bmatrix}\boldsymbol{K}_{ss} & \boldsymbol{G}_s\\ \boldsymbol{G}_s^T & \boldsymbol{0}\end{bmatrix}^{-1}\left(\begin{bmatrix}\boldsymbol{f}_s\\ \boldsymbol{0}\end{bmatrix} - \begin{bmatrix}\boldsymbol{K}_{s\Gamma}\\ \boldsymbol{G}_{\Gamma p}^T\end{bmatrix}\boldsymbol{u}_\Gamma^k\right) + \boldsymbol{K}_{\Gamma\Gamma}\boldsymbol{u}_\Gamma^k \\
&\qquad = \boldsymbol{f}_\Gamma - [\boldsymbol{M}_{\Gamma D} \quad \boldsymbol{G}_{\Gamma\varphi}]\begin{bmatrix}\boldsymbol{M}_{DD} & \boldsymbol{G}_D\\ \boldsymbol{G}_D^T & \boldsymbol{0}\end{bmatrix}^{-1}\left(\begin{bmatrix}\boldsymbol{f}_D\\ \boldsymbol{0}\end{bmatrix} - \begin{bmatrix}\boldsymbol{A}_{D\Gamma}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{u}_\Gamma^{(k-1)}\right)
\end{aligned}$$

$$\begin{aligned}
&\left(\boldsymbol{K}_{\Gamma\Gamma} - [\boldsymbol{K}_{\Gamma s} \quad \boldsymbol{G}_{\Gamma p}]\begin{bmatrix}\boldsymbol{K}_{ss} & \boldsymbol{G}_s\\ \boldsymbol{G}_s^T & \boldsymbol{0}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{K}_{s\Gamma}\\ \boldsymbol{G}_{\Gamma p}^T\end{bmatrix}\right)\boldsymbol{u}_\Gamma^k \\
&\qquad = \boldsymbol{f}_\Gamma - [\boldsymbol{K}_{\Gamma s} \quad \boldsymbol{G}_{\Gamma p}]\begin{bmatrix}\boldsymbol{K}_{ss} & \boldsymbol{G}_s\\ \boldsymbol{G}_s^T & \boldsymbol{0}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{f}_s\\ \boldsymbol{0}\end{bmatrix} - [\boldsymbol{M}_{\Gamma D} \quad \boldsymbol{G}_{\Gamma\varphi}]\begin{bmatrix}\boldsymbol{M}_{DD} & \boldsymbol{G}_D\\ \boldsymbol{G}_D^T & \boldsymbol{0}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{f}_D\\ \boldsymbol{0}\end{bmatrix} \\
&\qquad + [\boldsymbol{M}_{\Gamma D} \quad \boldsymbol{G}_{\Gamma\varphi}]\begin{bmatrix}\boldsymbol{M}_{DD} & \boldsymbol{G}_D\\ \boldsymbol{G}_D^T & \boldsymbol{0}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{A}_{D\Gamma}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{u}_\Gamma^{(k-1)}
\end{aligned}$$

4. Monolithic and partitioned schemes in time

Consider the one-dimensional, transient, heat transfer equation:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f \ \ in\ [0,1]$$

$$u(x = 0, t) = 0$$

$$u(x = 1, t) = 0$$

$$u(x, t = 0) = 0$$

4.1. Discretize it using the finite element method (linear elements, element size h) for the discretization in space, and a BDF1 scheme for the discretization in time. Write down the weak form of the problem and the resulting matrix form of the problem, including the corresponding boundary integrals if necessary. Consider $\kappa = 1, f = 1, \delta t = 1$.

The variable u is discretized as

$$u(x,t) \approx u_h(x,t) = \sum_{j=0}^{n} u_j(t) N_j(x)$$

The derivative in time is:

$$\partial_t u(x,t) \approx \partial_t u_h(x,t) = \sum_{j=0}^{n} \partial_t u_j(t) N_j(x)$$

The weak form of the problem is:

$$\int_0^1 v_h \cdot (\partial_t u_h - \kappa \Delta u_h)\, dx = \int_0^1 v_h \cdot f\, dx$$

Integrating by parts:

$$\int_0^1 v_h \cdot \partial_t u_h\, dx + \int_0^1 \nabla v_h \kappa \nabla u_h\, dx - [v_h \kappa \nabla u_h]_0^1 = \int_0^1 v_h \cdot f\, dx$$

Substituting the numerical values and the discretization:

$$\int_0^1 N_i \cdot \sum_{j=0}^{n} \partial_t u_j N_j\ dx + \int_0^1 \nabla N_i \sum_{j=0}^{n} u_j(t) N_j(x)\, dx = \int_0^1 N_i\ dx\ for\ i = 1, \dots, n-1$$

The first and last node have been excluded from the equation because its value is prescribed and the boundary integral, in this case a 0-dimensional integral, is null because all BC are Dirichlet.

This result in the following linear system:

$$\boldsymbol{M} \partial_t \boldsymbol{u_h} + \boldsymbol{K} \boldsymbol{u_h} = \boldsymbol{f}$$

$$\boldsymbol{M}_{ij} = \int_0^1 N_i \cdot N_j\, dx$$

$$\boldsymbol{K}_{ij} = \int_0^1 \nabla N_i \cdot \nabla N_j\, dx$$

$$\boldsymbol{f}_i = \int_0^1 N_i \, dx$$

The discretization in time of BDF1 is stated as: $\partial_t \boldsymbol{u}_{n+1} = (\boldsymbol{u}_{n+1} - \boldsymbol{u}_n)/\delta t$

Substituting:

$$\boldsymbol{M}\partial_t \boldsymbol{u}_{\boldsymbol{h}}^{n+1} + \boldsymbol{K}\boldsymbol{u}_{\boldsymbol{h}}^{n+1} = \boldsymbol{f}$$

$$\frac{\boldsymbol{M}}{\delta t}(\boldsymbol{u}_{\boldsymbol{h}}^{n+1} - \boldsymbol{u}_{\boldsymbol{h}}^{n}) + \boldsymbol{K}\boldsymbol{u}_{\boldsymbol{h}}^{n+1} = \boldsymbol{f}$$

$$\left(\frac{\boldsymbol{M}}{\delta t} + \boldsymbol{K}\right)\boldsymbol{u}_{\boldsymbol{h}}^{n+1} = \boldsymbol{f} - \boldsymbol{K}\boldsymbol{u}_{\boldsymbol{h}}^{n}$$

4.2. Consider a domain decomposition approach for the previous problem. The left subdomain is composed of 2 elements $(h = 0.2)$, **while the right subdomain is composed of 3 elements** $(h = 0.2)$. **Show that, if a monolithic approach is adopted, no boundary integrals are required at the interface. From now on, we denote the values at the nodes of the mesh as** $u_0, u_1, u_2, u_3, u_4, u_5$. **The interface is at** u_2.

The equations for the 1st subdomain are:

$$\int_0^{0.4} v_{1h} \cdot \partial_t u_{1h} \, dx + \int_0^{0.4} \nabla v_{1h} \nabla u_{1h} \, dx - [v_{1h} \nabla u_{1h}]_0^{0.4} = \int_0^{0.4} v_{1h} \; dx$$

The equations for the 2nd subdomain are:

$$\int_{0.4}^{1} v_{2h} \cdot \partial_t u_{2h} \, dx + \int_{0.4}^{1} \nabla v_{2h} \nabla u_{2h} \, dx - [v_{2h} \nabla u_{2h}]_{0.4}^{1} = \int_{0.4}^{1} v_{2h} \; dx$$

The transmission conditions are:

- Continuity of the temperature:

$$u_{1h}(\Gamma) = u_{2h}(\Gamma)$$

- Continuity of fluxes:

$$\nabla u_{1h} - \nabla u_{2h} = 0$$

From the first equation and neglecting the integral boundary at the Dirichlet BC:

$$[v_{1h} \nabla u_{1h}]_{x=0.4} = \int_0^{0.4} v_{1h} \cdot \partial_t u_{1h} \, dx + \int_0^{0.4} \nabla v_{1h} \nabla u_{1h} \, dx - \int_0^{0.4} v_{1h} \; dx$$

From the second equation:

$$\int_{0.4}^{1} v_{2h} \cdot \partial_t u_{2h} \, dx + \int_{0.4}^{1} \nabla v_{2h} \nabla u_{2h} \, dx + [v_{2h} \nabla u_{2h}]_{x=0.4} = \int_{0.4}^{1} v_{2h} \; dx$$

Substituting:

$$\int_{0.4}^{1} v_{2h} \cdot \partial_t u_{2h}\, dx + \int_{0.4}^{1} \nabla v_{2h} \nabla u_{2h}\, dx + \int_{0}^{0.4} v_{1h} \cdot \partial_t u_{1h}\, dx + \int_{0}^{0.4} \nabla v_{1h} \nabla u_{1h}\, dx$$
$$= \int_{0}^{0.4} v_{1h}\ dx + \int_{0.4}^{1} v_{2h}\ dx$$

The original problem has been recovered without boundary integrals.

4.3. Obtain the algebraic form of the Dirichlet-to-Neumann operator (Steklov-Poincaré operator) for the left subdomain, departing from given values of u_i^n at time step n, and an interface value u_2^{n+1}.

The matrices for the left problem is:

$$\left(\frac{\boldsymbol{M}}{\delta t} + \boldsymbol{K}\right) \boldsymbol{u_h^{n+1}} = \boldsymbol{f} - \boldsymbol{K}\boldsymbol{u_h^n}$$

The global matrices are computed as:

$$\boldsymbol{M} = h \cdot \begin{bmatrix} 2/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$$

$$\boldsymbol{K} = \frac{1}{h} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{f} = h \cdot \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ \phi^{n+1} \end{bmatrix}$$

Where ϕ^{n+1} is obtained from the boundary integral and represents the flux out of the boundary. The vector of unknowns is:

$$\boldsymbol{u_h} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The resulting system is:

$$\left(\frac{h}{\delta t} \cdot \begin{bmatrix} 2/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} + \frac{1}{h} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\right) \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \end{bmatrix} = h \cdot \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} - \frac{1}{h} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^n \\ u_2^n \end{bmatrix} + \begin{bmatrix} 0 \\ \phi^{n+1} \end{bmatrix}$$

Imposing the u_2^{n+1} as a Dirichlet BC:

$$\left(\frac{h}{\delta t}\frac{2}{3} + \frac{2}{h}\right) u_1^{n+1} = h - \frac{2}{h} u_1^n + \frac{1}{h}(u_2^{n+1} + u_2^n)$$

It can be written as:

$$A_1 u_1^{n+1} = F_1 + C_1 u_1^n + -B_1 u_2^{n+1}$$

Now ϕ^{n+1} can be computed:

$$\phi^{n+1} = \frac{h}{\delta t}\left(\frac{1}{6} u_1^{n+1} + \frac{1}{3} u_2^{n+1}\right) + \frac{1}{h}(u_2^{n+1} - u_1^{n+1}) - \frac{1}{h}(u_2^n - u_1^n) - \frac{h}{2}$$

So with this, the Dirichlet-to-Neumann operator has been computed: given u_2^{n+1}, u_1^{n+1} is computed and then ϕ^{n+1}.

4.4. Obtain the algebraic form of the Neumann-to-Dirichlet operator for the right subdomain, departing from given values of u_1^n and an interface value for the fluxes $\phi^{n+1} = k\partial_x u^{n+1}$ at the coordinate of node 2.

The system of equations is:

$$\left(\frac{\boldsymbol{M}}{\delta t} + \boldsymbol{K}\right)\boldsymbol{u_h^{n+1}} = \boldsymbol{f} - \boldsymbol{K}\boldsymbol{u_h^n}$$

The matrices are:

$$\boldsymbol{M} = h \cdot \begin{bmatrix} 1/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 2/3 \end{bmatrix}$$

$$\boldsymbol{K} = \frac{1}{h} \cdot \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\boldsymbol{f} = h \cdot \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\phi^{n+1} \\ 0 \\ 0 \end{bmatrix}$$

Here, $\boldsymbol{u_h^{n+1}}$ can be computed. $\boldsymbol{u_h^{n+1}}$ is:

$$\boldsymbol{u_h^{n+1}} = \begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \end{bmatrix}$$

The system can be written as:

$$A_2 u_2^{n+1} = F_2 + C_2 u_1^n - B_2 u_1^{n+1}$$

4.5. Write down the iterative algorithm for a staggered approach applying Dirichlet boundary conditions at the interface to the left subdomain and Neumann boundary conditions at the interface for the right subdomain.

In a staggered approach, instead of solving the monolithic system as one global linear system, two local problems are solved using some approximation. The system to solve is:

$$A_1 u_1^{n+1} = F_1 + C_1 u_1^n + -B_1 \tilde{u}_2^{n+1}$$

$$A_2 u_2^{n+1} = F_2 + C_2 u_1^n - B_2 \tilde{u}_1^{n+1}$$

In a 1st order approximation:

$$\tilde{u}_i^{n+1} = u_i^n$$

In a 2nd order approximation:

$$\tilde{u}_i^{n+1} = 2u_i^n - u_i^{n-1}$$

4.6. Do the same for a substitution and an iteration by subdomains scheme.

The substitution scheme is equal of the staggered approach but instead of solving the two problems independently, first it is solved for u_1^{n+1}, then $\tilde{u}_1^{n+1}$ is not approximation but taken as the already computed value.

The iteration by subdomains scheme is similar to the subdomains scheme but before advancing into the new time step, the approximation is actualized with the computed one until convergence is achieved:

$$A_1 u_1^{n+1,i} = F_1 + C_1 u_1^n + -B_1 \tilde{u}_2^{n+1,i-1}$$

$$A_2 u_2^{n+1,i} = F_2 + C_2 u_1^n - B_2 \tilde{u}_1^{n+1,i}$$

4.7. Rewrite the algebraic system associated to the left subdomain (Dirichlet boundary conditions at the interface) using Nitche's method for applying the boundary conditions. How does the condition number of the resulting system of equations vary with the penalty parameter α?

The system to solve is still the same with the contribution of the Nitche's method:

$$\left(\frac{\boldsymbol{M}}{\delta t} + \boldsymbol{K} + \boldsymbol{N}\right) \boldsymbol{u}_{\boldsymbol{h}}^{n+1} = \boldsymbol{f} + \boldsymbol{f}_N - \boldsymbol{K}\boldsymbol{u}_{\boldsymbol{h}}^n$$

The system however, include the u_0 degree of freedom as the Dirichlet BC is imposed weakly:

$$\boldsymbol{u}_{\boldsymbol{h}} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$

The matrices are:

$$\boldsymbol{M} = h \cdot \begin{bmatrix} 1/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{bmatrix}$$

$$\boldsymbol{K} = \frac{1}{h} \cdot \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\boldsymbol{f} = h \cdot \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}$$

Now, the terms corresponding to the Nitche's method are added in the left hand side the following terms are added:

$$\alpha \frac{k}{h} (v_h, u_h)_\Gamma - k < \boldsymbol{n} \cdot \nabla v_h, u_h >_\Gamma$$

This result in the additional contribution that has to be summed:

$$\boldsymbol{N} = \begin{bmatrix} \alpha \frac{k}{h} - \frac{k}{h} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \frac{k}{h} - \frac{k}{h} \end{bmatrix}$$

$$\boldsymbol{f}_N = \begin{bmatrix} 0 \\ 0 \\ \left(\alpha \frac{k}{h} - \frac{k}{h}\right) \bar{u}_2 \end{bmatrix}$$

The α must be chosen large enough to ensure stability of the system. For engineering problems $\alpha = \mathcal{O}(10^6 \div 10^9)$. However, the larger α is the greater is the condition number of the resulting matrix. For this reason, in most of cases the matrix is ill-conditioned

5. Operator splitting techniques

Consider the one dimensional, transient, convection-diffusion equation:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} + a_x \frac{\partial u}{\partial x} = f \;\; in[0,1]$$
$$u(x = 0, t) = 0$$
$$u(x = 1, t) = 0$$
$$u(x, t = 0) = 0$$

With $\kappa = 1, a_x = 1, f = 1$

5.1. Discretize it in space using finite elements (3 elements) and in time (finite differences, BDF1). Solve the first step of the problem, writing the solution as a function of the time step size $\boldsymbol{\delta t}$.

The space discretization is:

$$\int_0^1 v(\partial_t u - \kappa \Delta u + a_x \nabla u)\, dx = \int_0^1 vf\, dx$$

Integrating by parts:

$$\int_0^1 -v\kappa\Delta u\, dx = \int_0^1 \nabla v \kappa \nabla u\, dx - [v\kappa\nabla u]_0^1$$

The discretized weak form is (the boundary integral vanishes as all BC are of Dirichlet type and no contributions are added to the vector force because the value of imposed values are 0).

$$\int_0^1 v_h \partial_t u_h\, dx + \int_0^1 \nabla v_h \kappa \nabla u_h\, dx + \int_0^1 v_h a_x \cdot \nabla u_h\, dx = \int_0^1 vf\, dx$$

The algebraic form is:

$$\boldsymbol{M}\partial_t \boldsymbol{u_h} + \boldsymbol{K}\boldsymbol{u_h} + \boldsymbol{C}\boldsymbol{u_h} = \boldsymbol{F}$$

The BDF1 discretization is $\partial_t \boldsymbol{u}_{n+1} = (\boldsymbol{u}_{n+1} - \boldsymbol{u}_n)/\delta t$. Resulting in

$$\frac{\boldsymbol{M}}{\delta t}(\boldsymbol{u_h^{n+1}} - \boldsymbol{u_h^{n}}) + \boldsymbol{K}\boldsymbol{u_h^{n+1}} + \boldsymbol{C}\boldsymbol{u_h^{n+1}} = \boldsymbol{F}^{n+1}$$

The matrices are obtained as:

$$\boldsymbol{M}_{ij} = \int_0^1 N_i N_j\, dx$$

$$\boldsymbol{M} = h \cdot \begin{bmatrix} 2/3 & 1/6 \\ 1/6 & 2/3 \end{bmatrix}$$

$$\boldsymbol{K}_{ij} = \int_0^1 \nabla N_i \nabla N_j\, dx$$

$$\boldsymbol{K} = \frac{1}{h} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\boldsymbol{C}_{ij} = \int_0^1 N_i \nabla N_j \, dx$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}$$

$$\boldsymbol{F}_i = \int_0^1 N_i \, dx$$

$$\boldsymbol{F} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The resulting system is:

$$\left(\frac{1}{3\delta t} \cdot \begin{bmatrix} 2/3 & 1/6 \\ 1/6 & 2/3 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}\right) \begin{bmatrix} \boldsymbol{u}_1^1 \\ \boldsymbol{u}_2^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3\delta t} \cdot \begin{bmatrix} 2/3 & 1/6 \\ 1/6 & 2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 + 2/(9\delta t) & -7/2 + 1/(18\delta t) \\ -5/2 + 1/(18\delta t) & 6 + 2/(9\delta t) \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^1 \\ \boldsymbol{u}_2^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The solution of the system is computed explicitly as:

$$\begin{bmatrix} \boldsymbol{u}_1^1 \\ \boldsymbol{u}_2^1 \end{bmatrix} = \frac{1}{\frac{5}{108\delta t^2} + \frac{8}{3\delta t} + \frac{109}{4}} \begin{bmatrix} 6 + 2/(9\delta t) & 7/2 - 1/(18\delta t) \\ 5/2 - 1/(18\delta t) & 6 + 2/(9\delta t) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{u}_1^1 \\ \boldsymbol{u}_2^1 \end{bmatrix} = \frac{1}{\frac{5}{108\delta t^2} + \frac{8}{3\delta t} + \frac{109}{4}} \begin{bmatrix} \frac{19}{2} + \frac{1}{6\delta t} \\ \frac{17}{2} + \frac{1}{6\delta t} \end{bmatrix}$$

5.2. Solve the same time step by using a first order operator splitting technique.

We will use a continuous splitting:

$$\frac{\partial u}{\partial t} + \mathcal{L}_D u + \mathcal{L}_C u = f \;\; in[0,1]$$

Where

$$\mathcal{L}_D u = -\kappa \Delta u$$

$$\mathcal{L}_C u = a_x \cdot \frac{\partial u}{\partial x}$$

First, given u^n, only the convection operator will be applied to obtain u_C^{n+1}:

$$\frac{\partial u_C}{\partial t} + \mathcal{L}_C u_C = 0$$

Then, given u_C^{n+1} the solution of the next time step is computed by using the convection operator:

$$\frac{\partial u}{\partial t} + \mathcal{L}_D u = f$$

Discretizing the convection operator:

$$\frac{\boldsymbol{M}}{\delta t}\left(\boldsymbol{u}_C^{n+1}-\boldsymbol{u}^n\right)+\boldsymbol{C}\boldsymbol{u}_C^{n+1}=0 \rightarrow \left(\frac{\boldsymbol{M}}{\delta t}+\boldsymbol{C}\right)\boldsymbol{u}_C^{n+1}=\frac{\boldsymbol{M}}{\delta t}\boldsymbol{u}^n$$

As $\boldsymbol{u}^0=\boldsymbol{0}, \boldsymbol{u}_C^1=\boldsymbol{0}$

Now the diffusion term is added:

$$\frac{\boldsymbol{M}}{\delta t}\left(\boldsymbol{u}^{n+1}-\boldsymbol{u}_C^{n+1}\right)+\boldsymbol{K}\boldsymbol{u}^{n+1}=\boldsymbol{F}^{n+1} \rightarrow \left(\frac{\boldsymbol{M}}{\delta t}+\boldsymbol{K}\right)\boldsymbol{u}^{n+1}=\boldsymbol{F}^{n+1}+\frac{\boldsymbol{M}}{\delta t}\boldsymbol{u}_C^{n+1}$$

Substituting the previously computed values:

$$\left(\frac{1}{3\delta t}\cdot\begin{bmatrix}2/3 & 1/6\\ 1/6 & 2/3\end{bmatrix}+3\cdot\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\right)\boldsymbol{u}^1=\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}2/(9\delta t)+6 & 1/(18\delta t)-3\\ 1/(18\delta t)-3 & 2/(9\delta t)+6\end{bmatrix}\boldsymbol{u}^1=\begin{bmatrix}1\\1\end{bmatrix}$$

Solving the system:

$$\boldsymbol{u}^1=\frac{1}{\frac{5}{108\delta t^2}+\frac{3}{\delta t}+27}\begin{bmatrix}6+2/(9\delta t) & 3-1/(18\delta t)\\ 3-1/(18\delta t) & 6+2/(9\delta t)\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\boldsymbol{u}^1=\frac{1}{\frac{5}{108\delta t^2}+\frac{3}{\delta t}+27}\begin{bmatrix}9+1/(6\delta t)\\ 9+1/(6\delta t)\end{bmatrix}$$

5.3. Evaluate the error of the splitting approach with respect to the monolithic approach. Plot splitting error vs. time step size for $\boldsymbol{\delta t=1,\delta=0.5,\delta=0.25}$. Comment on the results.

The solution of the monolithic scheme is:

$$\begin{bmatrix}\boldsymbol{u^m}_1^1\\ \boldsymbol{u^m}_2^1\end{bmatrix}=\frac{1}{\frac{5}{108\delta t^2}+\frac{8}{3\delta t}+\frac{109}{4}}\begin{bmatrix}\frac{19}{2}+1/(6\delta t)\\ \frac{17}{2}+1/(6\delta t)\end{bmatrix}$$

The solution of the splitting scheme is:

$$\begin{bmatrix}\boldsymbol{u^s}_1^1\\ \boldsymbol{u^s}_2^1\end{bmatrix}=\frac{1}{\frac{5}{108\delta t^2}+\frac{3}{\delta t}+27}\begin{bmatrix}9+1/(6\delta t)\\ 9+1/(6\delta t)\end{bmatrix}$$

The values for the different values of δt have been substituted. To plot the error, it has computed the error of the first component u_1^1:

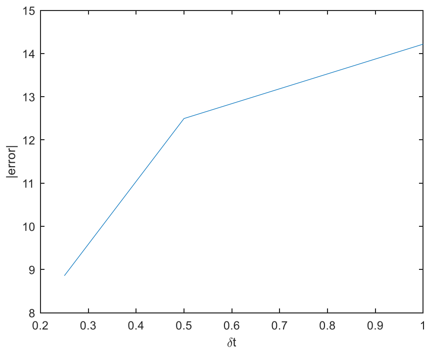

Figure 1: Convergence plot for splitting scheme

It is seen that convergence is ensured.

6. Transmission conditions

Consider the fractional step approach for the incompressible Navier-Stokes equations (Yosida scheme):

$$M\frac{1}{\delta t}\left(\widehat{U}^{n+1}-U^{n}\right)+K\widehat{U}^{n+1}=f-G\tilde{P}^{n+1}$$

$$DM^{-1}GP^{n+1}=\frac{1}{\delta t}D\widehat{U}^{n+1}-DM^{-1}G\tilde{P}^{n+1}$$

$$M\frac{1}{\delta t}\left(U^{n+1}-\widehat{U}^{n+1}\right)+\alpha K\left(U^{n+1}-\widehat{U}^{n+1}\right)+G\left(P^{n+1}-\tilde{P}^{n+1}\right)=0$$

6.1. Which is the optimal value for the α parameter?

To guess the optimal value for α in terms of accuracy, the 1st and 3rd equations are summed and it is obtained:

$$M\frac{1}{\delta t}(U^{n+1}-U^{n})+K\left(\alpha U^{n+1}+(1-\alpha)\widehat{U}^{n+1}\right)+\sigma\left(U^{n+1}-\widehat{U}^{n+1}\right)+G(P^{n+1})=f$$

Here it is seen that if $\alpha=1$ the original momentum equation is obtained with errors of order $\sigma\left(U^{n+1}-\widehat{U}^{n+1}\right)$ as the convection term is non-linear

6.2. What is the source of error of the scheme?

It has been already seen that the first source of error is that the convection term is nonlinear.

However, the largest source of error is in the imposition of the incompressibility condition. The Yosida method calculates first $\widehat{U}^{n+1}$ as a velocity only satisfying the momentum equation. Then, to enforce incompressibility, at 2nd equation it is computed an estimation of the gradient of the pressure needed to ensure that continuity is ensured $\left(P^{n+1}-\tilde{P}^{n+1}\right)$. And with the 3rd equation this is added as some type of "penalty" and U^{n+1} is computed as a correction of $\widehat{U}^{n+1}$ with a better approximation of the continuity condition.

7. ALE formulations

7.1. Given the spatial description of a property

$$\gamma(x,y,z,t)=[2x,ye^{t},z]$$

The equations of movement:

$$x=Xe^{t}$$
$$y=Y+e^{t}-1$$

$$z = Z$$

And the equations of the movement of the mesh:

$$x_m = X_m + \alpha t$$
$$y_m = Y_m - \beta t$$
$$z_m = Z_m$$

a) **Obtain the description of the property in terms of the ALE coordinates** (X_m, Y_m, Z_m).

$$\gamma_{ALE}(X_m, Y_m, Z_m, t) = [2(X_m + \alpha t), (Y_m - \beta t)e^t, Z_m]$$

b) **Compute the velocity of the particles and the mesh velocity**.

The velocity of the particles in term of the material coordinates is:

$$v_p(X,t) = \frac{\partial \Phi(\mathrm{X},\mathrm{t})}{\partial t} = \begin{bmatrix} Xe^t \\ e^t \\ 0 \end{bmatrix}$$

To obtain it in terms of spatial coordinates only the transformation Φ^{-1} is needed:

$$v_p(x,t) = \begin{bmatrix} x \\ e^t \\ 0 \end{bmatrix}$$

The velocity of the mesh is independent of material or spatial coordinates and only depends on time:

$$v_{mesh}(t) = \begin{bmatrix} \alpha \\ -\beta \\ 0 \end{bmatrix}$$

c) **Compute the ALE description of the material temporal derivative of γ.**

First, the gradient of γ is computed:

$$\nabla\gamma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now the derivative of γ_{ALE} is computed:

$$\frac{\partial \gamma_{ALE}}{\partial t} = \begin{bmatrix} 2\alpha \\ -\beta \\ 0 \end{bmatrix}$$

The relative velocity $v - v_m$ must be computed in terms of X_m, Y_m, Z_m for that the inverse of the equations of movement are used:

$$v - v_m = \begin{bmatrix} xe^{-t} - \alpha \\ e^t + \beta \\ 0 \end{bmatrix} = \begin{bmatrix} X_m + \alpha(t-1) \\ e^t + \beta \\ 0 \end{bmatrix}$$

Finally, the ALE material derivative is computed:

$$\frac{d\gamma_{ALE}}{dt} = \frac{\partial \gamma_{ALE}}{dt} + (v - v_m)\cdot\nabla\gamma = \begin{bmatrix} 2\alpha \\ -\beta \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} X_m + \alpha(t-1) \\ e^t + \beta \\ 0 \end{bmatrix} = \begin{bmatrix} 2X_m + 2\alpha t \\ \beta(e^t - 1) + e^{2t} \\ 0 \end{bmatrix}$$

7.2. Write down the ALE form of the incompressible Navier-Stokes equations. Where (in time and space) is each of the terms of the equation evaluated? How are temporal derivatives computed?

The Navier-Stokes for incompressible flow are:

$$\rho \frac{Du}{Dt} = \mu \nabla^2 u(\boldsymbol{x}, t) - \nabla p(\boldsymbol{x}, t) + \rho \boldsymbol{b}(\boldsymbol{x}, t)$$

$$\nabla \cdot \boldsymbol{u}(\boldsymbol{x}, t) = 0$$

The incompressibility condition remains unchanged. In the momentum balance equation, the material derivative is replaced by its ALE description:

$$\rho \frac{\partial u_{ALE}(\boldsymbol{X_m}, t)}{\partial t} + (v - v_m) \cdot \nabla \boldsymbol{u}(\boldsymbol{x}, t) = \mu \nabla^2 u(\boldsymbol{x}, t) - \nabla p(\boldsymbol{x}, t) + \rho \boldsymbol{b}(\boldsymbol{x}, t)$$

Most of the terms are evaluated in the Eulerian frame of reference. The difference is in the velocity of the temporal derivative. This u_{ALE} is evaluated always at the same nodes even if they are displaced. This is used in order to discretize the temporal derivatives using a finite difference method.

7.3. Do a bibliographical research on existing methods for the definition of the mesh movement in ALE formulations (Poisson problem, Elasticity problem, etc.). Describe the main advantages of each of these methods.

The boundaries of the meshes using ALE formulations must be able to be defined in some parts within a Eulerian reference system and within a Lagrangian one in other regions. They must also ensure that the mesh is not highly distorted.

The problem can be stated as: Given the displacement of the mesh in the boundary, find a map that fulfils this boundary condition (of Dirichlet type) and does not distort the mesh too much. Several techniques can be used:

- **Laplacian mesh**

In this method it is enforced that each component of the displacement is harmonic:

$$\nabla^2 d_i \; for \; i = 1, \dots, n_d$$

The problem is that for large displacements of the boundary this can result in self-intersection.

- **Elasticity problem**

Another approach with a physical base is to suppose that the domain behaves as an elastic body. Given the displacements on the boundary, the interior displacements are solved as part of a linear elasticity problem. This method ensures better results than the Laplacian but the computational cost is higher as the displacements in different directions are now coupled. However, for large displacements this method is not good as it is well known that applying infinitesimal strain theory to large deformations can result in self-intersection.

- **Transfinite mapping method**

This method was designed as mesh generator. It consists in mapping a reference domain into the actual domain. It can be applied at each time step and the topology will remain the same. In fact, any mesh generator that consists in a mapping from a reference domain can be used.

- **Mesh-smoothing**

In ALE algorithms it is possible to use any mesh-smoothing algorithm if they conserve the topology. This ensures a reduction of the deformation of the shape of the mesh.

References:

[1] *Arbitrary Lagrangian-Eulerian Methods.* J. Donea, A. Huerta, J.-Ph. Ponthot and A. Rodríguez-Ferran

8. Fluid-Structure Interaction

8.1. Describe the added mass effect problem for fluid structure interaction problems. When does it appear, what kind of problems suffer from it? What are the main methods for dealing with it?

The added mass is a phenomenon that appears when the fluid and solid density are similar. If this happens convergence of classical partitioned schemes. To fix this problem, methods that use relaxation offer good results.

A widely used method is the Aitken relaxation scheme. This method uses an adaptive factor of relaxation for a faster convergence.

8.2. Consider the iteration by subdomain scheme for the heat transfer problem described in problem 1. Apply 2 iterations of the AITKEN relaxation scheme to it.

The problem to solve is:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = f \;\; in\ [0,1]$$

$$u(x=0,t)=0$$

$$u(x=1,t)=0$$

$$u(x,t=0)=0$$

In problem 1 of section 4 we showed that the discretization of the problem (using BDF1) is:

$$\left(\frac{1}{\delta t}\boldsymbol{M} + \boldsymbol{K}\right)\boldsymbol{u}^{n+1} = \boldsymbol{f} + \frac{1}{\delta t}\boldsymbol{M}\boldsymbol{u}^{n}$$

The first domain is solved using Neumann BC at the interface:

$$\left(\frac{1}{\delta t}\boldsymbol{M}_1 + \boldsymbol{K}_1\right)\boldsymbol{u}_1^{n+1,k} = \boldsymbol{f}_1 + \frac{1}{\delta t}\boldsymbol{M}_1\boldsymbol{u}_1^{n}$$

Where $\boldsymbol{f}_1$ takes the contribution of the Neumann BC:

$$\frac{\partial u_1^{n+1,k}}{\partial x}(\Gamma) = -\frac{\partial u_2^{n+1,k-1}}{\partial x}(\Gamma)$$

The Dirichlet problem is:

$$\left(\frac{1}{\delta t}\boldsymbol{M}_2 + \boldsymbol{K}_2\right)\boldsymbol{u^*}_2^{n+1,k} = \boldsymbol{f}_2 + \frac{1}{\delta t}\boldsymbol{M}_2\boldsymbol{u}_2^{n}$$

The condition on Γ is:

$$u_2^{n+1,k}(\Gamma) = \omega\, u_1^{n+1,k}(\Gamma) + (1-\omega) u{*}_2^{n+1,k-1}(\Gamma)$$

Where the relaxation factor is defined as:

$$\omega = \frac{u_2^{n+1,k-2}(\Gamma) - u_2^{n+1,k-1}(\Gamma)}{u_2^{n+1,k-2}(\Gamma) - u_2^{n+1,k-1}(\Gamma) + u{*}_2^{n+1,k}(\Gamma) - u{*}_2^{n+1,k-1}(\Gamma)}$$

It is seen that in the two first iterations this formula is no valid and a fixed value of ω must be chosen. For that reason, the two first iterations of the Aitken scheme are the same as a classical partitioned scheme.

8.3. Consider the monolithic (1 domain), transient (BDF1), finite element (linear elements $h = 1/4$) approximation of the heat transfer equation in problem 1. Enforce Dirichlet boundary conditions in $x = 0$ and $x = 1$ by using Lagrange multipliers. What is the form of the discrete system? What is the condition number of the resulting matrix?

The elemental stiffness and mass matrices are:

$$\boldsymbol{K}^e = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{M}^e = h\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$$

After the assembly the global matrices are obtained:

$$\boldsymbol{K} = \begin{bmatrix} 4 & -4 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}$$

$$\boldsymbol{M} = \frac{1}{24}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The Lagrange multipliers are defined only at the boundaries:

$$\mu_1(0) = 1, \mu_1(1) = 0$$

$$\mu_2(0) = 0, \mu_2(1) = 1$$

The resulting system is:

$$\boldsymbol{Lu} = \boldsymbol{b}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The discretized system is:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{L}^T \\ \boldsymbol{L} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{n+1} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} + \frac{1}{\delta t}\boldsymbol{M}\boldsymbol{u}^n \\ \boldsymbol{b} \end{bmatrix}$$

Where:

$$\boldsymbol{A} = \frac{1}{\delta t}\boldsymbol{M} + \boldsymbol{K}$$

Assuming $\delta t = 1$, the condition number of the resulting system is of 38.32. The condition number is so large because of the presence of the Lagrange multipliers. Although the equations of $\boldsymbol{L}$ are exactly the same as applying the Dirichlet BCs by row and column elimination, as they are dimensionless, the condition number can be very large. The condition number of the matrix resulting from rows and columns elimination is of 5.36.

8.4. Consider the monolithic (1 domain), transient (BDF1), finite element (linear elements, $h = 1/4$) approximation of the heat equation in problem 1. Suppose that a level set function ($\psi = 0$ at $x = 0.4$) divides the domain into a high thermal conductivity ($\kappa = 100$) subdomain ($x \in [0,0.4]$) and a low thermal conductivity ($\kappa = 1$) subdomain ($x \in (0.4,1]$). Build the system matrix for this problem. Take into account the need for subintegrating the element cut by the level set function.

The elemental mass matrices remain unchanged. The elemental stiffness matrices for the 1st,3rd and 4th elements are:

$$\boldsymbol{K}^e = \frac{\kappa}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Where κ is constant along the element.

On the 2nd element:

$$\boldsymbol{K}^e = \int_{\frac{1}{4}}^{\frac{1}{2}} \begin{bmatrix} \nabla N_1^e \kappa \nabla N_1^e & \nabla N_1^e \kappa \nabla N_2^e \\ \nabla N_2^e \kappa \nabla N_1^e & \nabla N_2^e \kappa \nabla N_2^e \end{bmatrix} dx$$

$$= \int_{0.25}^{0.4} \begin{bmatrix} \nabla N_1^e \nabla N_1^e & \nabla N_1^e \nabla N_2^e \\ \nabla N_2^e \nabla N_1^e & \nabla N_2^e \nabla N_2^e \end{bmatrix} dx + \int_{0.4}^{0.5} 100 \cdot \begin{bmatrix} \nabla N_1^e \nabla N_1^e & \nabla N_1^e \nabla N_2^e \\ \nabla N_2^e \nabla N_1^e & \nabla N_2^e \nabla N_2^e \end{bmatrix} dx$$

$$= 100 \begin{bmatrix} 4 \cdot 4 & 4 \cdot (-4) \\ (-4) \cdot 4 & (-4) \cdot (-4) \end{bmatrix} \cdot (0.4 - 0.25) + \begin{bmatrix} 4 \cdot 4 & 4 \cdot (-4) \\ (-4) \cdot 4 & (-4) \cdot (-4) \end{bmatrix} \cdot (0.5 - 0.4)$$

$$\boldsymbol{K}^e = 241.6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The assembly of $\boldsymbol{K}$ leads to:

$$\boldsymbol{K} = \begin{bmatrix} 400 & -400 & 0 & 0 & 0 \\ -400 & 641.6 & -241.6 & 0 & 0 \\ 0 & -241.6 & 245.6 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}$$