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Assignment of Coupled Problem — Seyed Mohammadreza Attar Seyedi

1 Transmission Conditions

① Assuming for example that the beam is clamped at $x=0$ and $x=L$ the principle of virtual work (PTV) states that the solution $v(x)$ satisfies

$$EI\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} = \int_0^L \delta v f \qquad \delta v(0)=\delta v(L)=0 \quad \frac{d\delta v}{dx}(0)=\frac{d\delta v}{dx}(L)=0$$

a) Postulate the space of functions where both v and δv must belong

b) If $[0,L]=[0,P]\cup[P,L]$ obtain the transmisson conditions at P implied by regularity requirments.

c) obtain the transmission conditions at P that follow by imposing in the PTV that the integral is additive.

a) $$EI\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} < \infty \quad \in L^2$$

$$\int_0^L \delta v f < \infty$$

the second derivatives in H^2 $\qquad \delta v, v \in H^2$

b)

$$\int_{P-a}^{P+a}\frac{du^\varepsilon}{dx} = \int_{P-a}^{P-\varepsilon/2}\frac{dv}{dx} + \int_{P-\varepsilon/2}^{P+\varepsilon/2}\frac{dv^\varepsilon}{dx} + \int_{P+\varepsilon/2}^{P+a}\frac{dv}{dx}$$

$$\int_{P-a}^{P+a}\frac{dv^\varepsilon}{dx} = \int_{P-a}^{P-\varepsilon/2}\left(\frac{dv}{dx} + \varepsilon\left[\frac{v(P+\varepsilon/2)-v(P-\varepsilon/2)}{\varepsilon}\right]\right) + \int_{P+\varepsilon/2}^{P+a}\frac{dv}{dx}$$

$$\int_{P-a}^{P+a}\frac{dv^\varepsilon}{dx} = \int_{P-a}^{P+a}\left(\frac{dv}{dx} + \left[v(P^+)-v(P^-)\right]\right) + \int_{P+\varepsilon/2}^{P+a}\frac{dv}{dx}$$

$$\int_{P-a}^{P+a}\left(\frac{dv^\varepsilon}{dx}\right)^2 = \int_{P-a}^{P-\varepsilon/2}\left(\frac{dv}{dx}^2 + \varepsilon\left[\frac{(v(P+\varepsilon/2)-v(P-\varepsilon/2))^2}{\varepsilon^2}\right]\right) + \int_{P+\varepsilon/2}^{P+a}\left(\frac{dv}{dx}\right)^2$$

$$\int_{P-a}^{P+a}\left(\frac{d^2v^\varepsilon}{dx^2}\right)^2 = \int_{P-a}^{P-\varepsilon/2}\left(\frac{d^2v}{dx^2} + \varepsilon\left[\frac{(dv(P+\varepsilon/2)-dv(P-\varepsilon/2))^2}{\varepsilon^2}\right]\right) + \int_{P+\varepsilon/2}^{P+a}\left(\frac{d^2v}{dx^2}\right)^2$$

$$dv(P+\varepsilon/2) - dv(P-\varepsilon/2) = dv(P^+) - dv(P^-) = 0 \quad \xleftarrow{\varepsilon\to 0}$$

C)

$$(EI)_1 \int_0^{P} \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} + (EI)_2 \int_{P}^{L} \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} = \int_0^L \delta v f$$

$$(EI)_1 \left[\left(\frac{d\delta v}{dx}\frac{d^2v(P_-)}{dx^2}\right) - \int_0^{P_-} \frac{d\delta v}{dx}\frac{d^3 v}{dx^3}\right] + (EI)_2 \left[\left(\frac{d\delta v}{dx}\frac{-d^2v(P_+)}{dx^2}\right) - \int_{P_-}^{L} \frac{d\delta v}{dx}\frac{d^3 v}{dx^3}\right]$$

$$(EI)_1 \left[\left(\delta v \frac{d^3v(P_-)}{dx^3}\right) - \int_0^{P_-} \delta v \frac{d^4 v}{dx^4}\right] + (EI)_2 \left[\left(\delta v \frac{-d^3v(P_+)}{dx^3}\right) - \int_{P_-}^{L} \delta v \frac{d^4 v}{dx^4}\right]$$

$$(EI)_1 \left(\frac{d^3v(P_-)}{dx^3}\right) - (EI)_2\left(\frac{d^3v(P_+)}{dx^3}\right) = 0$$

② The maxwell Problem

$\nu \nabla \times \nabla \times u = f \quad \in \Omega$

$\nabla \cdot u = 0 \quad \in \Omega$

$n \times u = 0 \quad \text{on } \Omega$

a) write a variational statement of the Problem. Postulate the space function u must belong.

b) If Γ is a surface that intersect Ω. transmission condition across Γ

c) obtain transmisson condition Γ in variational form of Problem.

a) $$\int_\Omega (\nu\nabla\times\nabla\times u)\cdot v = \int_\Omega f v \Rightarrow \int_\Omega \nu\nabla\times u \cdot \nabla\times v - \int_{\partial\Omega} (\nu\nabla\times u \times n)\cdot v = \int_\Omega f v$$

we know that $n \times u = 0$ then $\Rightarrow \int_\Omega \nu\nabla\times u\cdot\nabla\times v = \int_\Omega f v$

b) the first transmission condition for maxwell equation $V_\Gamma = 0$

C) $$\int_\Omega \nu\nabla\times u\cdot\nabla\times v = \int_{\Omega_1} (\nu\nabla\times\nabla\times u)\cdot v + \int_{\partial\Omega_{12}} (\nu\nabla\times u\times n)\cdot v = \int_{\Omega_1} f v$$

$$\int_{\partial\Omega_{12}} (\nu\nabla\times u\times n)\cdot v - \int_{\partial\Omega_{21}} (\nu\nabla\times u\times n)\cdot v = 0$$

③ Navier equation for an elastic material

$$-2\mu\nabla\cdot(\varepsilon(u)) - \lambda\nabla(\nabla\cdot u) = \rho b$$

$$-\mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u) = \rho b$$

$$\mu\nabla\times(\nabla\times u) - (\lambda-2\mu)\nabla(\nabla\cdot u) = \rho b$$

a) write Varitional form of Pervious equation in the appropriate function space

$$2\mu\int_\Omega \nabla^s u : \nabla v - 2\mu\int_{\partial\Omega} \nabla^s u\cdot v - \lambda\int_\Omega \nabla u : \nabla v - \lambda\int_\Omega \nabla\cdot(\nabla u\cdot v) = \int_\Omega \rho b\cdot v$$

$$2\mu\int_\Omega \nabla^s u : \nabla v - 2\mu\int_{\partial\Omega} \nabla^s u\cdot v - \lambda\int_\Omega \nabla u : \nabla v - \lambda\int_\Omega (n\cdot\nabla u)\cdot v = \int_\Omega \rho b\cdot v$$

$$\mu\int_\Omega \nabla u : \nabla v - \mu\int_{\partial\Omega} \nabla u\cdot v - (\lambda+\mu)\int_\Omega \nabla u : \nabla v - (\lambda+\mu)\int_{\partial\Omega} (n\cdot\nabla u)\cdot v = \int_\Omega \rho b\cdot v$$

$$\mu\int_\Omega \nabla\times u\cdot\nabla\times v - \mu\int_\Omega (\nabla\times u\times n)\cdot v - (\lambda+2\mu)\int_\Omega \nabla u : \nabla v - (\lambda-2\mu)\int_{\partial\Omega} (n\cdot\nabla u)\cdot v = \int_\Omega \rho b\cdot v$$

$\boxed{2}$ Domain decomposition methods

① Consider Problem 1 of transmission Condition. Let $[0,L] = [0,L_1][L_2,L]$

a) Iteration by subdomain scheme based on Schwarz additive domain decomposition.

b) matrix version of Previous scheme has been discretized using finite elements.

Subdomain Ω_1

$$(EI)_1\int_0^{L_1} \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} = \int_0^{L_1} \delta v f$$

$V(0) = 0 \quad x=0$

$\frac{dv(0)}{dx} = 0 \quad x=0$

$V_1^{(k)} = V_2^{(k-1)} \quad \Gamma_{12}$

Subdomain Ω_2

$$(EI)_2\int_{L_2}^{L} \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} = \int_{L_2}^{L} \delta v f$$

$V(0) = 0 \quad x=L$

$\frac{dv(0)}{dx} = 0 \quad x=L$

$V_2^{(k)} = V_1^{(k-1)} \quad \Gamma_{21}$

$$\frac{dv_1^{(k)}}{dx} = \frac{dv_2^{(k-1)}}{dn} \quad \Gamma_{12} \qquad \frac{dv_2^{(k)}}{dn} = \frac{dv_1^{(k-1)}}{dn}$$

$$(EI)_1 \frac{d^2 v_1^{(k)}}{dn^2} = (EI)_2 \frac{dv_2^{2(k-1)}}{dn^2} \quad \Gamma_{12} \qquad (EI)_2 \frac{d^2 v_2^{(k)}}{dn^2} = (EI)_1 \frac{d^2 v_1^{(k-1)}}{dn^2}$$

$$(EI)_1 \frac{d^3 v_1^{(k)}}{dn^3} = (EI)_2 \frac{d^3 v_2^{(k-1)}}{dn^3} \quad \Gamma_{12} \qquad (EI)_2 \frac{d^3 v_2^{(k)}}{dn^3} = (EI)_1 \frac{d^3 v_1^{(k-1)}}{dn^3}$$

b)

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_i^{(k)} \\ u_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} b_i \\ b_\Gamma \end{bmatrix}$$

$$A = \sum \frac{d^2 \delta v_j}{dn^2} \frac{d^2 \delta v_i}{dn^2} \qquad b = \sum \delta v_j f \qquad u = [u_j]$$

$$A_{11} u_1^{(k)} = b_1 - A_{1\Gamma} u_\Gamma^{(k-1)} \qquad A_{22} u_2^{(k)} = b_2 - A_{2\Gamma} u_\Gamma^{(k-1)}$$

$u_\Gamma^{(k)} \longrightarrow$ overlapping interface region $\qquad u_i^{(k)} \longrightarrow$ the interior nodes

2. Consider Problem 2 of Section 1. Let Γ surface that intersects Ω

a) write down an iteration by subdomain scheme based on Dirichlet-Neumann

b) the expression the Steklov-Poincare operator

c) the matrix version of the previous scheme once space discretized using finite elements.

Subdomain Ω_1 $\qquad$ Subdomain Ω_2

$$\int_{\Omega_1} \nu \nabla \times u . \nabla \times v = \int_{\Omega_1} f v \qquad \int_{\Omega_2} \nu \nabla \times u . \nabla \times v = \int_{\Omega_2} f v u_2^{(k)}$$

$$\int_{\Gamma_{12}} (\nu \nabla \times u^{(k)} \times n) . v = \int_{\Gamma_{21}} (\nu \nabla \times u^{(k-1)} \times n) . v \quad \Gamma_{12} \qquad u^{(k)} = \bar{u} \quad \Omega_2$$

$L = k \longrightarrow$ Gauss seidel $\qquad L = k-1 \longrightarrow$ Jacobi b)

$$\Phi \rightarrow \int_{\partial\Omega_{12}} (\nabla \times \tilde{u} \times n) \cdot v - \int_{\partial\Omega_{21}} (\nabla \times \tilde{u} \times n) \cdot v$$

c)

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{(k)} \\ u_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_\Gamma - A_{\Gamma 2} u_2^{(k-1)} - A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(k-1)} \end{bmatrix}$$

$$A_{22} u_2^{(k)} = b_\Gamma - A_{2\Gamma} u_\Gamma^{(L)}$$

3 $-K \Delta u = f$ in Ω

$u = 0$ on $\partial\Omega$

a) Iteration by subdomain scheme based on the Dirichlet-Robin coupling

b) matrix version the previous scheme has been discretized using finite elements.

c) Schur complement as discrete version of the Steklov-Poincaré operator

d) Preconditioner for Schur complement equation from iterative scheme of section 1

a)

weak form $\quad -k \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v$

Dirichlet - Robin

Subdomain 1

$$-k \int_{\Omega_1} \nabla u_1 \cdot \nabla v = \int_{\Omega_1} f v$$

$$u_1^{(k+1)} = u_2^{(k)}$$

subdomain 2

$$-k \int_{\Omega_2} \nabla u_2 \cdot \nabla v = \int_{\Omega_2} f v$$

$$\nabla u_2^{(k+1)} + \gamma_2 u_2^{(k+1)} = \nabla u_1^{(L)} + \gamma_1 u_1^{(L)}$$

$L = k \rightarrow$ Gauss-Seidel $\qquad L = k-1 \rightarrow$ Jacobi

b)

Subdomain 1

$$u_1^{k+1} = A_{11}^{-1} (f_1 - A_{1\Gamma} u_\Gamma^k)$$

Subdomain 2

$$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_2^{k+1} \\ u_\Gamma^{k+1} \end{bmatrix} = \begin{bmatrix} f_2 \\ f_\Gamma - A_{\Gamma 2} u_1^{(k)} - (A_{\Gamma 2} + A_{\Gamma\Gamma}) u_\Gamma^k \end{bmatrix}$$

c) $S = A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - (A_{\Gamma 2} + A_{\Gamma\Gamma}) A_{22}^{-1} (A_{2\Gamma} + A_{\Gamma\Gamma})$

$S_1 = A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}$

$S_2 = A_{\Gamma\Gamma} - (A_{\Gamma 2} + A_{\Gamma\Gamma}) A_{22}^{-1} (A_{2\Gamma} + A_{\Gamma\Gamma})$

d) $u_1^k = A_{11}^{-1}(f_1 - A_{1\Gamma} u_\Gamma^k)$ $\quad$ $u_2^{(k)} = A_{22}^{-1}(f_2 - A_{2\Gamma} u_\Gamma^k)$

$\boxed{3}$ Coupling of heterogeneous Problem

① Consider beam described Problem 1 Section 1. an elastic wall that occupies the square $[0,L] \times [-L,0]$ where $y=0$ corresponds to the beam axis.

a) Write equation in wall assuming Plane stress behavior.

b) the beam modified because of the Presence of the wall.

c) the adequate transmisson and normal component of traction on the wall $y=0$

d) the tangent component on the wall $y=0$. Discuss the implication if component is not zero.

a)

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}$$

c)

Transmisson condition at $y=0$ is equal to $\sigma(u_{beam}) \cdot n$

d). the tangent component at $y=0$ we have it in x direction for tangent component if we have in x, y displacment we have Dirichlet transmisson.

② Let S_D and S_S be the Dirichlet-to-Neumann operators for the Darcy and Stokes Problem

a) discretized using finite elements. Resulting matrices the discretization of the Darcy and Stoke Problem

b) Dirichlet - Neumann iteration by subdomain using of Darcy and Stokes Problem.

a)

Stoke Problem : $-\nu \Delta u_S + \nabla P_S = f$ $\qquad$ $\nabla \cdot u_S = 0$

Darcy Problem : $u_D + k \nabla \phi = 0$ $\qquad$ $\nabla \cdot u_D = 0$

$$-\sum_{ij} \int_\Omega \left(\nu \nabla (\delta u_S)_i : \nabla (\delta u_S)_j - (P_S)_i \nabla \cdot (\delta u_S)_j \right)$$

$$-\sum \int \left((\delta u_s) \cdot (n \cdot -p_s I + \nu \nabla (\delta u_s)_j (u_s)_j) \right)$$

b)

$$\begin{bmatrix} A_{II} & B_{I\Gamma} & A_{I\Gamma} & 0 & 0 \\ B_{\Gamma I} & 0 & B_{I\Gamma} & 0 & 0 \\ A_{\Gamma I} & B_{\Gamma I} & A_{\Gamma\Gamma} & M_{\Gamma\Gamma} & 0 \\ 0 & 0 & -M_{\Gamma\Gamma} & A_{\Gamma\Gamma} & A_{I\Gamma} \\ 0 & 0 & 0 & A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_\Omega \\ p \\ u_\Gamma \\ \phi_\Gamma \\ \phi_\Omega \end{bmatrix}$$

$\boxed{4}$ Monolithic and Partitioned schemes in time

Consider the one dimensional, transient, heat transfer equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f \quad \text{in } [0,1] \qquad u(x=0,t)=0,\ u(x=1,t)=0,\ u(x,t=0)=0$$

a) Discretize using finite element and BDF1 scheme for discretiztion in time. write weak form of Problem and resulting matrix from of Problem. including the corresPonding boundry integrals. Consider $k=1$, $f=1$, $\delta t=1$

b) Consider domain decomposition approach for Previous Problem. The left subdomain ComPosed of 2 elements while right subdomain is composed 3 elements) we denote the value of meshes u_1, u_2, u_3 ...

c) obtain the Dirichlet-Neumann for left subdomain values u_i^n at time n and interface value u_2^{n+1}

d) obtain Neumann to Dirichlet operator for right subdomain and interface value for flux $\phi = k \partial x u^{n+1}$

e) iterative algorithm for staggered approach applying Dirichlet boundry Condition left and Neumann right

f) Calculated fluxes at the interface values from left substituted known values to right domain there is no need of Dirichlet-Neumann Prediction for temPerature at right domain

g) Rewrite the algebraic system to left subdomain using Nitsche's method for applying boundry

How does Condition number of resulting system of equation with Penalty Parameter α?

a) $$\int v\frac{\partial u}{\partial t} + \int k\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial x}\Big|_0^1 = \int vf$$

$$\frac{\partial u}{\partial t} = \frac{u^{n+1}-u^n}{\delta t} \Rightarrow \sum \int v_i (v_j u_j^{n+1} - v_j u_j^n) + \sum \int \frac{\partial v_i}{\partial x}\frac{\partial v_j u_j^{n+1}}{\partial x} - \sum \int v_i \frac{\partial v_j u_j^{n+1}}{\partial x} = \sum \int v_i$$

b) $\frac{\partial v_j u_j^{n+1}}{\partial n} - \sum \int_{\Gamma_{12}} v_i \frac{\partial v_j u_j^{n+1}}{\partial n} + \frac{\partial v_j u_j^{n+1}}{\partial n} - \sum \int_{\Gamma_{21}} v_i \frac{\partial v_j u_j^{n+1}}{\partial n} = 0$

c) $S(u) = \frac{\partial u_2^{n+1}}{\partial n} = \frac{u_2^{n+1} - (u_1^n + \frac{\partial u_1^n}{\partial t}\delta t)}{h}$

d)

$S(\phi) = u_3^{n+1} - \phi^{n+1} h = (u_3^n + \frac{\partial u_3^n}{\partial t}\delta t) - \phi^{n+1} h$

e)

$\sum \int v_i (v_j u_j^{n+1} - v_j u_j^n) + \sum \int \frac{\partial v_i}{\partial n} \frac{\partial v_j u_j^{n+1}}{\partial n} = \sum \int v_i f$

$\frac{\partial u_2}{\partial t}$ = Dirichlet to Neumann

$\sum \int v_i (v_j u_j^{n+1} - v_j u_j^n) + \sum \int \frac{\partial v_i}{\partial n} \frac{\partial v_j u_j^{n+1}}{\partial n} = \sum \int v_i f$

u_2^{n+1} = Neumann to Dirichlet

f)

$\sum_{ij} \int_{\Omega^e} v_i (v_j u_j^{n+1} - v_j u_j^n) + \sum \int_{\Omega^e} \frac{\partial v_i}{\partial n} \frac{\partial v_j u_j^{n+1}}{\partial n} = \sum \int_{\Omega} v_i f + \sum \int_{\partial\Omega} v_i \frac{\partial v_j \bar{u}_j^{n+1}}{\partial n} + \sum v_i v_j u_j^n$

$(u_2^{n+1}{}_L)^i = (u_2^{n+1}{}_R)^{i-1}$

$\sum \int_{\Omega^e} v_i (v_j u_j^{n+1} - v_j u_j^n) + \sum \int_{\Omega^e} \frac{\partial v_i}{\partial n} \frac{\partial v_j u_j^{n+1}}{\partial n} = \sum \int v_i f + \sum \int_{\partial\Omega} v_i \frac{\partial v_j \bar{u}_j^{n+1}}{\partial n} + \sum v_i v_j u_j^n$ (1a)

$\left(\frac{\partial u_2}{\partial n}{}_R\right)^i = \left(\frac{\partial u_1}{\partial n}{}_L\right)^i$

g)

$\sum_{ij} \int_{\Omega^e} v_i (v_j u_j^{n+1} - v_j u_j^n) + \sum \int \frac{\partial v_i}{\partial n} \frac{\partial v_j u_j^{n+1}}{\partial n} - \sum \int \frac{\partial v_i}{\partial n} \frac{\partial v_j u_j^{n+1}}{\partial n} - \sum \int v_i \frac{\partial v_j u_j^{n+1}}{\partial n} + \alpha \sum \int v_i v_j u_j$

$- \frac{\alpha k}{h} \sum \int \frac{\partial v_i}{\partial n} v_j u_j = \sum \int v_i + \alpha \sum \int v_i v_j \bar{u}_j - \frac{\alpha k}{h} \sum \int_{\Gamma} \frac{\partial v_i}{\partial n} v_j \bar{u}_j$

5 operator splitting techniques

1) Consider one dimensional, transient, convection diffusion: $\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + a_x \frac{\partial u}{\partial x} = f$

$k=0,\ a_x=1,\ f=1.$

a) Discretize using Finite element. Solve first step of the problem, writing function of time step δt

b) Solve the same time step by using first order operator splitting technique.

c) Evaluate Error of the splitting approach. Plot splitting Error vs time step size $t = 1, 0.5, 0.25$

a)
$$\int_0^1 v\frac{\partial u}{\partial x} + \int_0^1 k\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial x}\Big|_0^1 - a_x\int_0^1 v\frac{\partial u}{\partial x} = \int_0^1 vf$$

$$\int_0^1 v_i \frac{v_j(u_j^{n+1} - u_j^n)}{\delta t} + \int_0^1 k\frac{\partial v_i}{\partial x}\frac{\partial v_j u_j}{\partial x} - v_j\frac{\partial v_j u_j}{\partial x}\Big|_0^1 - a_x\int_0^1 v_i\frac{\partial v_j u_j}{\partial x} = \int_0^1 v_i f$$

$$k_{11}^e = \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(\frac{x^2}{3} - x_2^e x + (x_2^e)^2\right) + kx + a_x x\left(x_2^e - \frac{x}{2}\right)\right]_{x_1}^{x_2}$$

$$k_{12}^e = \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(x_1x_2 - \frac{x_1x_2}{2} - \frac{x_2 x}{2} + \frac{x^2}{3}\right) - kx + a_x x\left(x/2 - x_1\right)\right]_{x_1}^{x_2}$$

$$k_{21}^e = \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(x_1x_2 - \frac{x_1 x}{2} - \frac{x_2 x}{2} + \frac{x^2}{3}\right) - kx - a_x x\left(x/2 - x_2\right)\right]_{x_1}^{x_2}$$

$$k_{22}^e = \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(\frac{x^2}{3} - x_1^e x + (x_1^e)^2\right) + kx + a_x x\left(x/2 - x_1\right)\right]_{x_1}^{x_2}$$

$$f_1^e = \left[fx\left(x_2 - \frac{x}{2}\right)\right]_{x_1}^{x_2} + \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(\frac{x^2}{3} - x_2^e x + (x_2^e)^2\right)\right]u_1^n + \frac{1}{l^e}\left[\frac{x}{\delta t}\left(x_1x_2 - \frac{x_1 x}{2} - \frac{x_2 x}{2} + \frac{x^2}{3}\right)\right]u_2^n$$

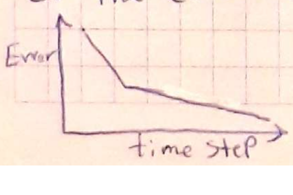
$$f_2^e = \left[fx\left(x/2 - x_1\right)\right]_{x_1}^{x_2} + \frac{1}{(l^e)^2}\left[\frac{x}{\delta t}\left(x_1x_2 - \frac{x_1 x}{2} - \frac{x_2 x}{2} + \frac{x^2}{3}\right)\right]u_1^n + \frac{1}{l^e}\left[\frac{x}{\delta t}\left(\frac{x^2}{3} - x_1^e x + (x_1^e)^2\right)\right]u_2^n$$

$$\begin{bmatrix} \frac{1}{9\delta t} + \frac{7}{2} & -\frac{1}{18\delta t} - \frac{5}{2} \\ -\frac{1}{18\delta t} - \frac{5}{2} & \frac{1}{9\delta t} + \frac{7}{2} \end{bmatrix}\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{f}{18} \\ \frac{f}{18} \end{bmatrix} + \begin{bmatrix} \frac{1}{9\delta t} & -\frac{1}{18\delta t} \\ -\frac{1}{18\delta t} & \frac{1}{9\delta t} \end{bmatrix}\begin{bmatrix} u_1^n \\ u_2^n \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{9\delta t} + 7/2 & \frac{-1}{18\delta t} - 5/2 & 0 & 0 \\ \frac{-1}{18\delta t} - 5/2 & \frac{1}{9\delta t} + 7/2 & \frac{-1}{18\delta t} + 5/2 & 0 \\ 0 & \frac{1}{18\delta t} - 5/2 & \frac{1}{9\delta t} + 7/2 & \frac{-1}{18\delta t} + 5/2 \\ 0 & 0 & \frac{-1}{18\delta t} + 7/2 & \frac{1}{9\delta t} + 7/2 \end{bmatrix}\begin{bmatrix} 0 \\ u_1^{n+1} \\ u_2^{n+1} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{9} \\ \frac{1}{9} \\ \frac{1}{18} \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{9\delta t} + 7 & \frac{-1}{18\delta t} - 5/2 \\ \frac{-1}{18\delta t} - 5/2 & \frac{1}{9\delta t} + 7/2 \end{bmatrix}\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \end{bmatrix} = \begin{bmatrix} 1/9 \\ 1/9 \end{bmatrix}$$

c) the error become larger when the time step decrease



6 Fractional step method

1. Consider fractional step approach for the incompressible Navier Stokes equation:

$$M\frac{1}{\delta t}(\hat{u}^{n+1}-u^{n})+K\hat{u}^{n+1}=f-G\hat{p}^{n+1}$$

$$DM^{-1}Gp^{n+1}=\frac{1}{\delta t}D\hat{u}^{n+1}-DM^{-1}G\hat{p}^{n-1}$$

$$M\frac{1}{\delta t}(u^{n+1}-\hat{u}^{n+1})+\alpha K(u^{n+1}-\hat{u}^{n+1})+G(p^{n+1}-\tilde{p}^{n+1})=0$$

a) which is the optimal value for α parameter?

b) what is the sarce of error of the scheme?

a). The optimal value for α is $2/3$ we need for solution in $n+1, n$ as we know BDF2 is third order in time and the best method for this case.

b). the error ocuur when we want to solve the velocity at pressure time step $n+1$ because the factoristion is prediction in pressure we have error at the end

7 ALE formulation

1. the spatial description of property $\gamma(x,y,z,t)=[2x, ye^{t}, z]$

equation of movement: $x=Xe^{t}$, $y=Y+e^{t}-1$, $z=Z$

the movement of the mesh: $x_m=\chi+\alpha t$, $y_m=\mathcal{Y}-\beta t$, $z_m=\mathcal{Z}$

a) obtain the property in term of the ALE coordinates $(\chi,\mathcal{Y},\mathcal{Z})$.

b) compute the velocity of the particles and the mesh velocity.

a)

$$\gamma(X,Y,Z,t)=[2Xe^{t}, Ye^{t}+e^{2t}-e^{t}, Z]$$

$$\gamma(\chi,\mathcal{Y},\mathcal{Z},t)=[2(\chi+\alpha t)e^{t}, (\mathcal{Y}-\beta t)e^{t}+e^{2t}-e^{t}, \mathcal{Z}]$$

b)

velocity of the mesh $\frac{\partial x_m}{\partial t}=\alpha$, $\frac{\partial y_m}{\partial t}=-\beta$, $\frac{\partial z_m}{\partial t}=0$

velocity of particle $\frac{\partial x(X)}{\partial t}=Xe^{t}$, $\frac{\partial y(X)}{\partial t}=e^{t}$, $\frac{\partial z(X)}{\partial t}=0$

② Write ALE from the incompressible Navier Stokes equation. Where in time and space is each of the term of the equation evaluated? How are temporal derivatives computed?

$$\frac{\partial u_x}{\partial t} + [v - v_{mesh} \cdot \nabla] u_x - \nu \Delta P_x u_x + \nabla P_x = f_x$$

$$\nabla \cdot u_x = 0$$

③ the definition of the mesh movement in ALE formulations. Describe the main advantages of each of these methods.

The shape of boundry may chenge after moving domain and transmisson condition have to be imposed between subdomain and applying right boundry in each moment of time is important. In noninear solid for changing shape we need ALE formulaction and for contact problem of soid for deformation and stress states we need ALE formultion. the ALE formulation the displacment and deformation have important role in this formulation

8 Fluid - structure interaction

① Describe the added mass effect problem for fluid structure interaction problems. When does it appear, what kind of problems suffer from it? What are the main methods for dealing with it?

The mass effect problem appear when the density of incompressible fluid solid near to each other and coupled problem does not converge.

for solving two subdomain we use the lumped mass matrix.

when the value of the unknown is imposed as boundry Condition the known quantity go to right hand side vector

the Convergence of the iterative depend on the relaxtion between densities of the fluid and solid has to satisfies an inequality.