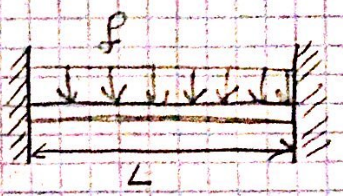
Aram Khaloian Coupled problems

Transmission Condition:



1. $EI\dfrac{d^4 v}{dx^4} = f$, PTR: $EI\displaystyle\int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2} = \int_0^L \delta v f$

$\rightarrow$ with: $\delta v(0) = \delta v(L) = 0$, $\dfrac{d\delta v}{dx}(0) = \dfrac{d\delta v}{dx}(L) = 0$

a) Because the two integrals should be bounded (no $\lim \int \rightarrow \infty$) then $v, \delta v$ should satisfy the $L^2(\Omega)$ space where:

$$L^2(\Omega) = \left\{ u(x) : \int_\Omega u^2 < \infty \right\}$$

Also both $v, \delta v$ should satisfy the space of functions whose derivatives are square integrable or $H^1(\Omega)$ where:

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : |\nabla u| \in L^2(\Omega) \right\}$$

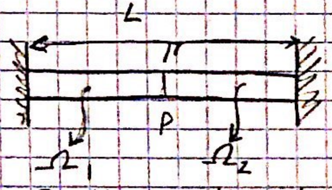
we know: $\nabla\cdot(\delta v \nabla \delta v) = \nabla\delta v \cdot \nabla \delta v + \delta v\, \nabla\cdot(\nabla \delta v) \longrightarrow$

$\rightarrow \displaystyle\int_\Omega \nabla\cdot(\delta v \nabla\delta v)\, d\Omega = \int_\Omega (\nabla\delta v\cdot\nabla\delta v)\, d\Omega + \int_\Omega \delta v\, \nabla\cdot(\nabla\delta v)\, d\Omega \rightarrow$ Considering a 1D domain

$\rightarrow \underbrace{\delta v \cancel{\dfrac{d\delta v}{dx}}\Big|_\Gamma}_{\text{on the boundary is zero}} = \displaystyle\int_\Omega \left(\frac{d\delta v}{dx}\right)^2 dx + \int_\Omega \delta v \frac{d^2\delta v}{dx^2}\, dx \rightarrow \int_\Omega \left(\frac{d\delta v}{dx}\right)^2 dx = -\int_\Omega \delta v \frac{d^2\delta v}{dx^2}\, d\Omega$

So $\longrightarrow \displaystyle\int_\Omega |\nabla v|^2 = -\int_\Omega \delta v\, \nabla\cdot\nabla v\, d\Omega \rightarrow \boxed{|\nabla v|^2 \in L^2}$

b) In case of $[0,L] = [0,P] \cup (P,L]$



in case of a discontinuity in our domain like P then the trasmission conditions should apply at the discontinuity:

$$[\![v]\!] = 0 \longrightarrow \lim_{\varepsilon\to 0} v(x_0 + \varepsilon n_1) - v(x_0 + \varepsilon n_2) = 0$$

$$[\![\nabla v]\!] = 0$$

c) By integrating by parts we have:

$$EI\left(\delta v \frac{d^3 v}{dx^3}\Big|_0^L - \frac{d\delta v}{dx}\frac{d^2 v}{dx^2}\Big|_0^L + \int_0^L \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\, d\Omega \right) = \int_0^L \delta v f$$

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for : Ω_1 : $EI\left(\delta v \frac{d^3 v}{dx^3}\Big|_0^{P_1} - \frac{d\delta v}{dx}\frac{d^2 v}{dx^2}\Big|_0^{P_1} + \int_0^{P_1} \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\, d\Omega_1 \right) = \int_0^{P_1} \delta v f \, d\Omega_1$

Ω_2 : $EI\left(-\delta v \frac{d^3 v}{dx^3}\Big|_{P_2}^{L} + \frac{d\delta v}{dx}\frac{d^2 v}{dx^2}\Big|_{P_2}^{L} + \int_{P_2}^{L} \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\, d\Omega_2 \right) = \int_{P_2}^{L} \delta v f \, d\Omega_2$

because $\Omega = \Omega_1 \cup \Omega_2$ then:

$$EI\left(\delta v \frac{d^3 v}{dx^3}\Big|_0^{P_1} - \delta v \frac{d^3 v}{dx^3}\Big|_{P_2}^{L} - \frac{d\delta v}{dx}\frac{d^2 v}{dx^2}\Big|_0^{P_1} + \frac{d\delta v}{dx}\frac{d^2 v}{dx^2}\Big|_{P_2}^{L} + \int_0^{P_1} \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\, d\Omega_1 + \int_{P_2}^{L} \frac{d^2\delta v}{dx^2}\frac{d^2 v}{dx^2}\, d\Omega_2 \right) = \int_0^{P_1} \delta v f \, d\Omega_1 + \int_{P_2}^{L} \delta v f \, d\Omega_2$$

we get: $\left(\delta v \frac{d^3 v}{dx^3}\Big|_0^{P_1} - \delta v \frac{d^3 v}{dx^3}\Big|_{P_2}^{L} \right) = 0$, $\left(\frac{d\delta v}{dx}\frac{d^2 v}{dx^2}\Big|_{P_2}^{L} - \frac{d\delta v}{dx}\frac{d^2 v}{dx^2}\Big|_0^{P_1} \right) = 0$

because $P_1 = P_2 \rightarrow$ all the integrals should be one

at point P: $\frac{d^2 v}{dx^2}\Big|_{P_1} = \frac{d^2 v}{dx^2}\Big|_{P_2} \rightarrow$ so the bending moments are continuous

and the same for the shear force: $\frac{d^3 v}{dx^3}\Big|_{P_1} = \frac{d^3 v}{dx^3}\Big|_{P_2}$

2. Maxwell problem: $\begin{cases} \nu \nabla\times\nabla\times u = f & \text{in } \Omega \\ \nabla\cdot u = 0 & \text{in } \Omega \\ n\times u = 0 & \text{on } \partial\Omega \end{cases}$ if $\begin{cases} \nu > 0 \\ f \text{ divergence free} \end{cases}$

a) $\int_\Omega v\cdot(\nu\nabla\times\nabla\times u)\, d\Omega = \int_\Omega v\cdot f\, d\Omega$

$\rightarrow \int_\Omega v\cdot(\nu\nabla\times\nabla\times u)\, d\Omega = \int_\Omega \nu\nabla\cdot(\nabla\times u \times v) + \int_\Omega (\nabla\times u)\cdot(\nabla\times v)\, d\Omega \rightarrow$ Divergence Th.

$= \int_\Omega \nu n\cdot(\nabla\times u\times v)\, d\Omega + \int_\Omega (\nabla\times u)\cdot(\nabla\times v)\, d\Omega$

$\rightarrow \int_\Omega \nu(n\times\nabla\times u)\cdot v\, d\Omega + \int_\Omega \nabla\times u\cdot(\nabla\times v)\, d\Omega = \int_\Omega v\cdot f\, d\Omega$

for: $\forall v \in H_{curl} = \{ P: \Omega \rightarrow \mathbb{R}^3 / P \in L^2(\Omega); \nabla\times P \in L^2(\Omega) \}$

b) so it should: $\int_\Omega |\nabla\times P|^2 < \infty$ & $\int_\Gamma (\nabla\times P)\cdot n\, dS = \oint_L P\cdot t\, dl$ in stokes

then the transmission condition will be: $[\![n\times u]\!] = 0$ on Γ

c) We have: Ω_1: $\int_{\Gamma_{12}} \vartheta[n_1\times(\nabla\times u_1)]\cdot v\, d\Gamma_{12} + \int_{\partial\Omega_1} \vartheta[n_1\times(\nabla\times u_1)]\cdot v\, d\Omega_1$

$+ \int_{\Omega_1}(\nabla\times u_1)\cdot(\nabla\times v)\, d\Omega_1 = \int_{\Omega_1} v\cdot f_1\, d\Omega_1$

Ω_2: $\int_{\Gamma_{21}} \vartheta[n_2\times(\nabla\times u_2)]\cdot v\, d\Gamma_{21} + \int_{\partial\Omega_2} \vartheta[n_2\times(\nabla\times u_2)]\cdot v\, d\Omega_2$

$+ \int_{\Omega_2}(\nabla\times u_2)\cdot(\nabla\times v)\, d\Omega_2 = \int_{\Omega_2} v\cdot f_2\, d\Omega_2$

summing the two we'll have the original week form because:

1) $\Omega = \Omega_1 \cup \Omega_2 \rightarrow \int_{\Omega_1}(\nabla\times u_1)\cdot(\nabla\times v)\, d\Omega_1 + \int_{\Omega_2}(\nabla\times u_2)\cdot(\nabla\cdot v)\, d\Omega_2 = \int_{\Omega}(\nabla\times u)\cdot(\nabla\times v)\, d\Omega$

2) $\int_{\Gamma_{12}} \vartheta[n_1\times(\nabla\times u_1)]\cdot v\, d\Gamma_{12} + \int_{\Gamma_{21}} \vartheta[n_2\times(\nabla\times u_2)]\cdot v\, d\Gamma_{21} = 0$

3) $\int_{\partial\Omega_1} \vartheta[n_1\times(\nabla\times u_1)]\cdot v\, d\Omega_1 + \int_{\partial\Omega_2} \vartheta[n_2\times(\nabla\times u_2)]\cdot v\, d\Omega_2 = \int_{\partial\Omega} \vartheta[n\times(\nabla\times u)]\cdot v\, d\Omega$

so from three we gain the transmission condition along the interface

$[\![\rho\times n]\!] = 0 \longrightarrow \rho = \nabla\times u \rightarrow [\![\nabla\times u\times n]\!] = 0$

3. Navier-Stokes equations: $\begin{cases} -2\mu\nabla\cdot(\varepsilon(u)) - \lambda\nabla(\nabla\cdot u) = \rho b & ① \\ -\mu\Delta u - (\lambda+\mu)\nabla(\nabla\cdot u) = \rho b & ② \\ \mu\nabla\times(\nabla\times u) - (\lambda+2\mu)\nabla(\nabla\cdot u) = \rho b & ③ \end{cases}$

and $u = 0$ on $\partial\Omega$

a) ① $\int_{\Omega} -2\mu\, v\cdot(\nabla\cdot\nabla^S u) + \int_{\Omega} -\lambda v\cdot\nabla(\nabla\cdot u) = \int_{\Omega} v\cdot\rho b$ → integrating by parts the R.H.S

$\longrightarrow -2\mu\int_{\Gamma} v\cdot(\nabla^S u)\cdot n + 2\mu\int_{\Omega}\nabla v : \nabla^S u - \lambda\int_{\Gamma} v(\nabla\cdot u)\cdot n + \lambda\int_{\Omega}(\nabla\cdot v)(\nabla\cdot u) = \int_{\Omega} v\cdot\rho b$

② using the guass theorem and integration by parts:

$-\mu\int_{\Gamma} v\cdot(\nabla u)\cdot n + \mu\int_{\Omega}\nabla v : \nabla u - (\lambda+\mu)\int_{\Gamma} v(\nabla\cdot u)\cdot n + (\lambda+\mu)\int_{\Omega}(\nabla\cdot v)(\nabla\cdot u) = \int_{\Omega} v\cdot\rho b$

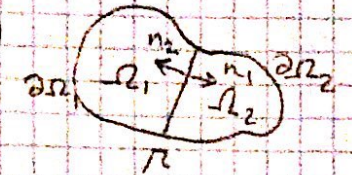
③ we know: $\nabla^2 u = \nabla(\nabla\cdot u) - \nabla\times\nabla\times u$ so:

using the above property equation 3 will be the same as 2

$\rightarrow -\mu\int_{\Gamma} v\cdot(\nabla u)\cdot n + \mu\int_{\Omega}\nabla v : \nabla u - (\lambda+\mu)\int_{\Gamma} v(\nabla\cdot u)\cdot n + (\lambda+\mu)\int_{\Omega}(\nabla\cdot v)(\nabla\cdot u) = \int_{\Omega} v\cdot\rho b$

as $\forall v \in H^1(\Omega)$ & $u \in H^1(\Omega)$

b)



$$\Omega_1 : -\mu\int_{\Gamma_{12}} v\cdot(\nabla u_1)\cdot n_1 + \mu\int_{\Omega_1}\nabla v : \nabla u - (\lambda+\mu)\int_{\Gamma_{12}} v\,(\nabla\cdot u_1)\cdot n_1 + (\lambda+\mu)\int_{\Omega_1}(\nabla\cdot v)(\nabla\cdot u) = \int_{\Omega_1} v\,\rho b_1$$

$$\Omega_2 : -\mu\int_{\Gamma_{21}} v\cdot(\nabla u_2)\cdot n_2 + \mu\int_{\Omega_2}\nabla v : \nabla u_2 - (\lambda+\mu)\int_{\Gamma_{21}} v\,(\nabla\cdot u_2)\cdot n_2 + (\lambda+\mu)\int_{\Omega_2}(\nabla\cdot v)(\nabla\cdot u) = \int_{\Omega_2} v\,\rho b_2$$

as indicated in the previous examples the sum on Ω_1 & Ω_2 will be the global term on Ω. So summing over the two expressions we will have:

$$\mu\int_{\Omega}\nabla v : \nabla u_2 + (\mu+\lambda)\int_{\Omega}(\nabla\cdot v)(\nabla\cdot u) - \mu\int_{\Gamma_{12}} v\cdot(\nabla u_1)\cdot n_1 - \mu\int_{\Gamma_{21}} v\cdot(\nabla u_2)\cdot n_2$$
$$- (\lambda+\mu)\int_{\Gamma_{12}} v\,(\nabla\cdot u_1)\cdot n_1 - (\lambda+\mu)\int_{\Gamma_{21}} v\,(\nabla\cdot u_2)\cdot n_2 = \int_{\Omega} v\cdot\rho b$$

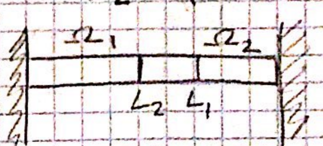
adding the four extra terms in comparison to the original weak form we will obtain the transmission conditions as: $[\![(\nabla u)\cdot n]\!] = 0$ & $[\![\nabla\cdot u]\!] = 0$

2. Domain Decomposition

1. $EI\frac{d^4u}{dx^4}=f$, $EI\int_0^L \frac{d^2\delta u}{dx^2}\frac{du^2}{dx^2}=\int_0^L \delta u f$ dom: $[0,L]=[0,L_1]\cup[L_2,L]$

$L_2<L_1$

a) Because the two subdomains are overlapping then the schwarz method should be used.

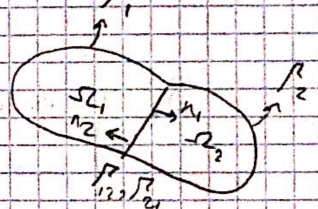


$$\Omega_1 : \begin{cases} EI\frac{d^4u_1^k}{dx^4}=f_1 & x\in[0,L_1] \\ u_1^{(k)}=0 & \text{on } x=0 \\ \frac{du_1^k}{dx}=0 & \text{on } x=0 \\ u_1^{(k)}=u_2^{(k-1)} & \text{on } x=L_1 \end{cases} \qquad \Omega_2 : \begin{cases} EI\frac{d^4u_2^k}{dx^4}=f_2 & x\in[L_2,L] \\ u_2^{(k)}=0 & \text{on } x=L \\ \frac{du_2^{(k)}}{dx}=0 & \text{on } x=L \\ u_2^{(k)}=u_1^{(k-1)} & \text{on } x=L_2 \end{cases}$$

b) $\begin{bmatrix} A_{11} & A_{1\Gamma_{12}} \\ A_{\Gamma_{12}1} & A_{\Gamma_{12}\Gamma_{12}} \end{bmatrix}\begin{bmatrix} u_1^{(k)} \\ u_{\Gamma_{12}} \end{bmatrix}=\begin{bmatrix} f_1 \\ f_{\Gamma_{12}} \end{bmatrix}$ for Ω_1 & $\begin{bmatrix} A_{22} & A_{2\Gamma_{21}} \\ A_{\Gamma_{21}2} & A_{\Gamma_{21}\Gamma_{21}} \end{bmatrix}\begin{bmatrix} u_2^{(k)} \\ u_{\Gamma_{21}} \end{bmatrix}=\begin{bmatrix} f_1 \\ f_{\Gamma_{21}} \end{bmatrix}$ on Ω_2

$\rightarrow u_{1\Gamma_{12}}=u_2^{(k-1)}$ & $u_{2\Gamma_{21}}=u_1^{(k-1)}$

2. Dirichlet-Neumann Coupling of Maxwell Prob.



a) $$\Omega_1 : \begin{cases} \nu\nabla\times\nabla\times u_1^{(k)}=f_1 & \text{on } \Omega_1 \\ \nabla\cdot u_1^{(k)}=0 & \text{on } \Omega_1 \\ n_1\times u_1^{(k)}=0 & \text{on } \Gamma_1 \\ n_1\times(\nabla\times u_1^{(k)})=n_1\times(\nabla\times u_2^{(k-1)}) & \text{on } \Gamma_{12} \end{cases}$$

$$\Omega_2 : \begin{cases} \nu\nabla\times\nabla\times u_2^{(k)}=f_2 & \text{on } \Omega_2 \\ \nabla\times u_2^{(k)}=0 & \text{on } \Omega_2 \\ n_2\times u_2^{(k)}=0 & \text{on } \Gamma_2 \\ n_2\times(\nabla\times u_2^{(k)})=n_2\times(\nabla\times u_1^{(k-1)}) & \text{on } \Gamma_{21} \end{cases}$$

b) For the Steklov Poincaré operator we split the solution as: $u=u_i^0+\tilde{u}_i$ and $i=1,2$

for every Ω_i: $$\begin{cases} \nu\nabla\times\nabla\times u_i^0=f_i & \text{on } \Omega_i \\ \nabla\cdot u_i^0=0 & \text{on } \Omega_i \\ n_i\times u_i^0=0 & \text{on } \Gamma_i \\ n_i\times u_i^0=0 & \text{on } \Gamma_{12} \end{cases} \quad \& \quad \begin{cases} \nu\nabla\times\nabla\times\tilde{u}_i=0 & \text{on } \Omega_i \\ \nabla\cdot\tilde{u}_i=0 & \text{on } \Omega_i \\ n_i\times\tilde{u}_i=0 & \text{on } \Gamma_i \\ n_i\times\tilde{u}_i=\varphi & \text{on } \Gamma_{12} \end{cases}$$

the unknown φ must satisfy the transmission condition.

$n_1\times(\nabla\times(u_1^0+\tilde{u}_1))=n_2\times(\nabla\times(u_2^0+\tilde{u}_2)) \rightarrow n_1\times(\nabla\times\tilde{u}_1)-n_2\times(\nabla\times\tilde{u}_2)=$
$n_1\times(\nabla\times u_2^0)-n_1\times(\nabla\times u_1^0)$

$\rightarrow \boxed{S\varphi=G}$

c) $\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} & 0 \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)}+A_{\Gamma\Gamma}^{(2)} & A_{\Gamma I}^{(2)} \\ 0 & A_{I\Gamma}^{(2)} & A_{II}^{(2)} \end{bmatrix} \begin{bmatrix} u_I^{(1)} \\ u_\Gamma \\ u_I^{(2)} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_\Gamma \\ f_2 \end{bmatrix}$

for the Ω_1 with Neumann boundary we will have:

$$\begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \begin{bmatrix} u_I^{(1)(k)} \\ u_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_\Gamma - A_{\Gamma I}^{(2)} u_I^{(2)(k-1)} - A_{\Gamma\Gamma}^{(2)} u_\Gamma^{(2)(k-1)} \end{bmatrix}$$

and for Ω_2 with Dirichlet boundary:

$$A_{II}^{(2)} u_I^{(2)(k)} = f_2 - A_{I\Gamma}^{(2)} u_\Gamma^{k-1}$$

3. $\begin{cases} -K\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ $K > 0$

a) Ω_1: $\begin{cases} -K\Delta u_1 = f_1 & \text{on } \Omega_1 \\ u_1 = 0 & \text{on } \partial\Omega_1 \\ u_1^{(k)} = u_2^{(i)} & \text{on } \Gamma \end{cases}$ Ω_2: $\begin{cases} -K\Delta u_2 = f_2 & \text{on } \Omega_2 \\ u_2 = 0 & \text{on } \partial\Omega_2 \\ K\frac{\partial u_2^{(k)}}{\partial n} + H_2 u_2^{(k)} = K\frac{\partial u_1^{(i)}}{\partial n} + H_1 u_1^{(i)} \end{cases}$

the value of i depends on the method used. In case jacobian $i = k-1$ and if Gauss $i = k$

for this $H_1 + H_2 > 0$ & $H_1 \neq H_2$

b) Ω_1: integration by parts $\rightarrow \int_{\Omega_1} K\nabla w_1 \nabla u_1 = \int_{\Omega_1} w_1 f_1$

Ω_2: $\int_{\Omega_2} K\nabla w_2 \nabla u_2 - \int_\Gamma K w_2 \nabla u_2 \cdot n = \int_{\Omega_2} w_2 f_2 \rightarrow$ using the expression for the flux of Robin

$$\rightarrow \int_{\Omega_2} K\nabla w_2 \nabla u_2 + \int_\Gamma K H_2 w_2 u_2 = \int_{\Omega_2} w_2 f_2 + \int_\Gamma K H_1 w_2 \frac{\partial u_1}{\partial n} + \int_{\Gamma_2} K H_2 w_2 u_1$$

Writing in matrix form: $\Omega_1: A_{II}^{(1)} u_I^{(1)(k)} = F_1 - A_{I\Gamma}^{(1)} u_\Gamma^{(k)}$
and Assembling

$$\Omega_2: \begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(1)} + H_2 M_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} + H_2 M_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} u_I^{(2)(k)} \\ u_\Gamma^{(k)} \end{bmatrix} = \begin{bmatrix} F_2 \\ F_\Gamma - \left[A_{\Gamma I}^{(1)} - H_1 M_{\Gamma I}^{(1)}\right] u_I^{(1)(k-1)} - (A_{\Gamma\Gamma}^{(1)} - H_1 M_{\Gamma\Gamma}^{(1)}) u_\Gamma^{(k-1)} \end{bmatrix}$$

c) Schur complement: $\Omega_1: u_I^{(1)(k)} = A_{II}^{(1)-1}(F_1 - A_{I\Gamma}^{(1)} u_\Gamma^{(1)(k)})$

$\Omega_2: u_I^{(2)(k)} = A_{II}^{(2)-1}(F_2 - A_{I\Gamma}^{(2)} u_\Gamma^{(k)})$

$\rightarrow \left[-A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)} + A_{\Gamma\Gamma}^{(2)}\right] u_\Gamma^{(k)} = F_\Gamma - A_{\Gamma I}^{(1)} u_I^{(1)(k-1)} - A_{\Gamma\Gamma}^{(1)} u_\Gamma^{k-1} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2$

using the term for Ω_1 in the above equation we will have:

$\left[-A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)} + A_{\Gamma\Gamma}^{(2)}\right] u_\Gamma^{(k)} = F_\Gamma - A_{\Gamma I}^{(1)} A_{II}^{(1)-1}(F_1 - A_{I\Gamma}^{(1)} u_\Gamma^{(k)}) - A_{\Gamma\Gamma}^{(1)} u_\Gamma^{k-1} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2$

Simplifying we will have:

$\left[A_{\Gamma\Gamma}^{(2)} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)} + A_{\Gamma\Gamma}^{(1)} - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} A_{I\Gamma}^{(1)}\right) u_\Gamma^k = F_\Gamma - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2 - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} F_1$

$\rightarrow S u_\Gamma = G$ the Schur complement

D) Changing the way of writing to $u_\Gamma^k = u_\Gamma^{k-1} + S^{-1}(F - S u_\Gamma^{k-1})$
and using the equation built i c we will have:

$S_1 = A_{\Gamma\Gamma}^{(1)} - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} A_{I\Gamma}^{(1)}$ & $S_2 = A_{\Gamma\Gamma}^{(2)} - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} A_{I\Gamma}^{(2)} \rightarrow S = S_1 + S_2$

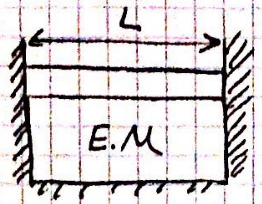
$G = F_\Gamma - A_{\Gamma I}^{(2)} A_{II}^{(2)-1} F_2 - A_{\Gamma I}^{(1)} A_{II}^{(1)-1} F_1$

$\rightarrow (S_1 + S_2) u_\Gamma = G \rightarrow S_2 u_\Gamma^k = G - S_1 u_\Gamma^{k-1} \rightarrow S_2 u_\Gamma^k = G - S u_\Gamma^{k-1} + S_2 u_\Gamma^{k-1}$

$\rightarrow u_\Gamma^k = u_\Gamma^{k-1} + S_2^{-1}(G - S u_\Gamma^{k-1})$

So S_2^{-1} is the preconditioner used.

3. Coupling of heterogeneous problems

1. 

a) From Hook's law we have:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\gamma_{xy} \end{bmatrix}$$

$$= \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{bmatrix}$$

we know: $\nabla \cdot \sigma + b = 0$

$$\rightarrow \begin{cases} \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 v}{\partial x \partial y} + \left(\frac{1-\nu}{2}\right)\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + b_x = 0 \\ \frac{\partial^2 v}{\partial y^2} + \nu \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{1-\nu}{2}\right)\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + b_y = 0 \end{cases}$$

b) Because of the wall an extra uniform load will be added to the force applied in the equation before.

$$EI \frac{d^4 v}{dx^4} = f \longrightarrow EI \frac{d^4 v}{dx^4} = f - h\sigma_y \qquad \text{where:}$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\nu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \rightarrow EI \frac{d^4 v}{dx^4} = f - h\left(\left(\frac{E}{1-\nu^2}\right)\left(\nu \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right)$$

c) In order to find the transmission condition for this problem we need to check the boundary and which parameters effect each other. The first one is the vertical displacement so:

$$[\![v]\!] = 0$$

The other one will be the traction between the two:

$$[\![n \cdot \nabla \cdot \sigma]\!] = 0$$

d) Because there is free slip condition between the two boundaries then the $[\![u]\!] \neq 0$.

Also the traction between the two won't be the same

$$[\![t \cdot \nabla \cdot \sigma]\!] \neq 0.$$

2. Stokes and Darcy regions interface

$$S_s(\lambda) = S_D(\lambda)$$

a) Stokes:

$$\begin{cases} -\mu\Delta u_s + \nabla p_s = f \\ \nabla\cdot u_s = 0 \end{cases}$$

Darcy:

$$\begin{cases} K^{-1} u_d + \nabla\varphi = 0 \\ \nabla\cdot u_d = 0 \end{cases}$$

weak form Stokes:

$$\int_{\Omega_s} \mu \nabla v_s : \nabla u_s \, d\Omega_s - \int_{\Omega_s} \nabla\cdot v_s \, p_s \, d\Omega_s - \int_{\partial\Omega_s} v_s\cdot\left[n\cdot(-p_s I + \nu\nabla^s u_s)\right] d\Omega = \int_{\Omega_s} v_s\cdot f \, d\Omega_s$$

weak form Darcy:

$$\int_{\Omega_D} v_D\cdot K^{-1} u_D \, d\Omega_D - \int_{\Omega_D} \nabla\cdot v_D \, \varphi \, d\Omega_D + \int_{\partial\Omega_D} v_D\cdot n\varphi = 0$$

$$\begin{cases} v_s = v_s^n\cdot n + v_s^t\cdot t \\ v_D = v_D^n\cdot n + v_D^t\cdot t \end{cases} \quad \text{so} \quad v_s^n = v_D^n$$

from the weak form of stokes:

$$v_s\cdot\left[n\cdot(-p_s I + \mu\nabla^s u_s)\right] = v_s^n p - \mu n\cdot\nabla^s u_s\cdot n v_s^n - \mu n\cdot\nabla^s u_s\cdot t v_s^t$$

from the weak form of Darcy: $v_D\cdot n\cdot\varphi I = v_D^n \varphi$

$$\rightarrow \begin{cases} v_s^n\left[p_s - \mu n\cdot\nabla^s u_s\cdot n\right] = v_D^n\varphi \rightarrow \varphi = p_s - \mu n\cdot\nabla^s u_s\cdot n \\ \mu(n\nabla^s u_s)\cdot t = -\frac{\alpha_{BS}}{\sqrt{K}}(u_s - u_D)\cdot t \end{cases}$$

$$\rightarrow \begin{cases} \int_{\Omega_s}\mu\nabla u_s:\nabla v \, d\Omega_s + \int_\Gamma v_s^t \frac{\alpha_{BS}}{\sqrt{K}}(u_s - u_D)\cdot t \, d\Gamma - \int_\Omega p_s\nabla\cdot v \, d\Omega - \int_\Gamma v_s^n n\left[-p_s I + \mu\nabla^s u_s\right] d\Gamma = \int_\Omega v\cdot f \, d\Omega \\ -\int_\Omega q_s \nabla\cdot u_s = 0 \end{cases}$$

$$\rightarrow \begin{cases} A^S_{II}u_I + A^S_{I\Gamma}u_\Gamma + B^{TS}p_{sI} = F^S_I & \text{on } \Omega_s \\ A^S_{\Gamma I}u_I + A^S_{\Gamma\Gamma}u_\Gamma + B^S_{I\Gamma}p_s + M^d_{\Gamma\Gamma}p_{d\Gamma} = F^S_\Gamma & \text{on } \partial\Omega_s \\ A^D_{II}p_{DI} + A^D_{I\Gamma}p_{D\Gamma} = F^D_I & \text{on } \partial\Omega_D \\ A^{TD}_{I\Gamma}p_{DI} + A^D_{\Gamma\Gamma}p_{D\Gamma} - M^{TD}_{\Gamma\Gamma}u_\Gamma = F^D_\Gamma & \text{on } \Gamma \\ B^S_{II}u_I + B^S_{I\Gamma}u_\Gamma = F^D_I & \text{on } \Omega_D \end{cases}$$

$$\begin{bmatrix} A^S_{II} & B^{TS} & A^S_{I\Gamma} & 0 & 0 & 0 \\ B^S_{II} & 0 & B^S_{I\Gamma} & 0 & 0 & 0 \\ A^S_{\Gamma I} & B^{TS}_{I\Gamma} & A^S_{\Gamma\Gamma} & M^D_{\Gamma\Gamma} & 0 & 0 \\ 0 & 0 & 0 & -M^{TD}_{\Gamma\Gamma} & A^D_{\Gamma\Gamma} & A^D_{I\Gamma} \\ 0 & 0 & 0 & 0 & A^D_{I\Gamma} & A^D_{II} \end{bmatrix} \begin{bmatrix} u_I \\ p_{sI} \\ u_\Gamma \\ p_{D\Gamma} \\ p_{DI} \end{bmatrix} = \begin{bmatrix} F^S_\Gamma \\ F_I \\ F^S_\Gamma \\ F^D_\Gamma \\ F^0_I \end{bmatrix}$$

b) Neumann $\rightarrow$ Darcy:

$$\begin{bmatrix} A^{D}_{\Gamma\Gamma} & A^{TD}_{I\Gamma} \\ A^{D}_{I\Gamma} & A^{D}_{II} \end{bmatrix} \begin{bmatrix} p^{D(k+1)}_{\Gamma} \\ p^{D(k+1)}_{I} \end{bmatrix} = \begin{bmatrix} F^{D}_{\Gamma} + M^{TD}_{\Gamma\Gamma} u^{(k)}_{\Gamma} \\ F^{D}_{I} \end{bmatrix}$$

Dirichlet $\rightarrow$ Stokes:

$$\begin{bmatrix} A^{S}_{II} & B^{TS}_{I} & A^{S}_{I\Gamma} \\ B^{S}_{II} & 0 & B^{S}_{I\Gamma} \\ A^{S}_{I\Gamma} & B^{TS}_{I\Gamma} & A^{S}_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_{I} \\ p_{SI} \\ u_{\Gamma} \end{bmatrix} = \begin{bmatrix} F^{S}_{I} \\ F_{I} \\ F^{S}_{\Gamma} - M^{D}_{\Gamma\Gamma} P^{\ell}_{D\Gamma} \end{bmatrix}$$

where $\ell = \begin{cases} \rightarrow k+1 & \text{Gauss Seidel} \\ \rightarrow k & \text{Jacobi} \end{cases}$

4. Monolithic and Partitioned schemes in time

1. One dimensional, transient, heat transfer:

$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} = f \quad \text{in } [0,1]$$

$$\begin{cases} u(x=0,t) = 0 \\ u(x=1,t) = \theta \\ u(x,t=0) = 0 \end{cases}$$

1) Using the θ family method: $\frac{\Delta u}{\Delta t} - \theta \Delta u_t = u_t^n$, $\begin{cases} \Delta u = u^{n+1} - u^n \\ \Delta u_t = u_t^{n+1} - u_t^n \end{cases}$

$$\longrightarrow \frac{\Delta u}{\Delta t} - \theta K \Delta(u_{xx}) = \theta f^{n+1} + 1 - \theta f^n + K u_{xx}^n$$

$$\longrightarrow \int w \frac{\Delta u}{\Delta t} d\Omega + \theta \int \frac{dw}{dx} K \frac{d\Delta u}{dx} d\Omega = \theta \int w f^{n+1} d\Omega + (1-\theta)\int w f^n d\Omega - \int \frac{dw}{dx}\frac{du^n}{dx} d\Omega$$

using $\theta = 1$: $(M + \Delta t K)\Delta u = \Delta t f^{n+1} - \Delta t K u^n$, $u^{n+1} = u^n + \Delta u$

where : $M_{ij} = \sum_e \int N_i N_j$, $K_{ij} = \sum_e \int \frac{dN_i}{dx}\frac{dN_j}{dx}$, $f_i = \sum_e \int N_i f$

2) Ω_1 : $\left(v_1, \frac{\partial u_1}{\partial t}\right)_{\Omega_1} + \left(\frac{\partial v_1}{\partial x}, \frac{\partial u_1}{\partial x}\right)_{\Omega_1} - \left(v_1, \frac{\partial u_1}{\partial x}\right)_{\Gamma} = (v_1, f_1)_{\Omega_1}$

Ω_2 : $\left(v_2, \frac{\partial u_2}{\partial t}\right)_{\Omega_2} + \left(\frac{\partial v_2}{\partial x}, \frac{\partial u_2}{\partial x}\right)_{\Omega_2} - \left(v_2, \frac{\partial u_2}{\partial x}\right)_{\Gamma} = (v_2, f_2)_{\Omega}$

while: $K\frac{\partial u_2}{\partial x} = -K\frac{\partial u_1}{\partial x} \rightarrow \left(v, K\frac{\partial u_2}{\partial x}\right)_{\Gamma} = -\left(v, K\frac{\partial u_1}{\partial x}\right)_{\Gamma}$

$$\rightarrow \left(v, \frac{\partial u_2}{\partial t}\right)_{\Omega_2} + \left(\frac{\partial v}{\partial x}, \frac{\partial u_2}{\partial x}\right)_{\Omega_2} + \left(v, \frac{\partial u_1}{\partial t}\right)_{\Omega_1} + \left(\frac{\partial v}{\partial x}, \frac{\partial u_1}{\partial x}\right)_{\Omega_1} = (v, f_2) + (v, f_1)$$

3) In matrix notation we'll have: $A\Delta u = f$

where: $A = \frac{M}{\Delta t} + K$, $f = \left(f + \frac{M}{\Delta t}\right)$

$$\rightarrow \begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} \end{bmatrix} \cdot \begin{bmatrix} \Delta u_I^{(1)} \\ \Delta u_\Gamma \end{bmatrix} = \begin{bmatrix} f_1 \\ f_\Gamma \end{bmatrix} \longrightarrow \begin{cases} \Delta u_I^{(1)} = u_I^{n+1} - u_I^n \\ \Delta u_\Gamma = u_\Gamma^{n+1} - u_\Gamma^n \end{cases}$$

$$\rightarrow \Delta u_I^{(1)} = A_{II}^{(1)-1} f_1 - A_{II}^{(1)-1} A_{I\Gamma}^{(1)} \Delta u_\Gamma$$

$$\rightarrow u_I^{n+1(1)} = A_{II}^{(1)-1} f_1 - A_{II}^{(1)-1} A_{I\Gamma}^{(1)} u_\Gamma^{n+1} + A_{II}^{(1)-1} A_{I\Gamma}^{(1)} u_\Gamma^{n(1)} + u_I^{n(1)}$$

4) $$\begin{bmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \Delta u_I^{(2)} \\ \Delta u_\Gamma^{(2)} \end{bmatrix} = \begin{bmatrix} f_2 \\ f_\Gamma^{(2)} + t \end{bmatrix}, \quad t = \int_\Gamma w K \frac{\partial u_\Gamma^{(1)n+1}}{\partial n}$$

$$\rightarrow U_\Gamma^{n+1} = f_\Gamma^{(1)} - A_{\Gamma\Gamma}^{(1)} u_\Gamma^{(1)n} - A_{\Gamma I}^{(1)} u_I^{(1)n+1}$$

5) $$\Omega_1: \begin{cases} \partial_t u_1^{k+1} - K \frac{\partial^2 u_1^{(k+1)}}{\partial x^2} = f_1 & \text{on } \Omega_1 \\ u_1^{k+1} = u_2^k & \text{on } \Gamma \\ u_1 = 0 & \text{on } \Gamma_D \end{cases} \qquad \Omega_2: \begin{cases} \partial_t u_2^{k+1} - K \frac{\partial^2 u_2^{k+1}}{\partial x^2} = f_2 & \text{on } \Omega_2 \\ K \frac{\partial u_2^{(k+1)}}{\partial n} = -K \frac{\partial u_1^{k+1}}{\partial n} & \text{on } \Gamma \\ u_2 = 0 & \text{on } \Gamma_D \end{cases}$$

6) $$\Omega_1: \begin{cases} \partial_t u_1^{k+1} - K \frac{\partial^2 u_1^{k+1}}{\partial n} = f_2 & \text{on } \Omega_1 \\ u_1^{k+1} = u_2^k & \text{on } \Gamma \\ u_1 = 0 & \text{on } \Gamma_D \end{cases} \qquad \Omega_2: \begin{cases} \partial_t u_2^{k+1} - K \frac{\partial^2 u_2^{k+1}}{\partial x^2} = f^2 & \text{on } \Omega_2 \\ K \frac{\partial u_2^{k+1}}{\partial n} = -K \frac{\partial u_1^{k+1}}{\partial n} & \text{on } \Gamma \\ u_2 = 0 & \text{on } \Gamma_D \end{cases}$$

7) $$\left(v_1, \frac{\partial u_1}{\partial x}\right) + (\nabla v, \nabla u) + \alpha (v_1, u_1)_\Gamma - (v_1 n \cdot K \nabla u_1)_\Gamma - (u_1, n \cdot K \nabla v_1)_\Gamma \Rightarrow$$

$$\rightarrow (v_1, f) + \alpha (v_1, u_D)_\Gamma - (u_D, n \cdot K \nabla v_1)_\Gamma$$

$$\rightarrow \begin{bmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} + \alpha M - B - B^T \end{bmatrix} \begin{bmatrix} \Delta u_I^{(1)} \\ \Delta u_\Gamma^{(1)} \end{bmatrix} = \begin{bmatrix} f_I^{(1)} \\ f_\Gamma^{(1)} + \alpha M u_D - B^T u_D \end{bmatrix}$$

where : $M = \int N^T N$, $B = \int N^T \frac{dN}{dn}$, $u^{n+1} = \Delta u + u^n$

In Nitsche's method the good thing is that it doesn't require a large number for α and that the left hand side matrix is symmetric-positive definite (S.P.D). But in case α increases largly with it the condition number is increased and we may encounter difficulties in the calculations.

5. Operator splitting techniques

$$\frac{\partial u}{\partial t} - K\frac{\partial^2 u}{\partial x^2} + a_x\frac{\partial u}{\partial x} = f \quad \text{on } [0,1]$$

$u(x=0,t) = 0$
$u(x=1,t) = 0$, $K=1, a_x=1, f=1$
$u(x,t=0) = 0$

1) Considering the theta method with $\theta = 1$

$$\frac{\Delta u}{\Delta t} + (a.\nabla - \nabla^2)\Delta u = f^{n+1} - (a.\nabla - \nabla^c)u^n$$

Discretizing in space : (Matrix form)

$$\left(\frac{M}{\Delta t} + C + K\right)u^{n+1} = f^{n+1} + \frac{M}{\Delta t}u^n$$

We have the shape functions as: $N_1 = \frac{1}{2}(1-\xi)$, $N_2 = \frac{1}{2}(1+\xi)$

$$M^e = \int_{-1}^{1}\frac{l}{2}\begin{bmatrix}\frac{1}{2}(1-\xi)\\ \frac{1}{2}(1+\xi)\end{bmatrix}\begin{bmatrix}\frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi)\end{bmatrix}d\xi = \begin{bmatrix}1/9 & 1/18\\ 1/18 & 1/9\end{bmatrix}$$

$$K^e = \int_{-1}^{1}\frac{l}{2}\begin{bmatrix}-\frac{1}{l^e}\\ \frac{1}{l^e}\end{bmatrix}\begin{bmatrix}-1/l^e & 1/l^e\end{bmatrix}d\xi = \begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$$

$$C^e = \int_{-1}^{1}\frac{l}{2}\begin{bmatrix}\frac{1}{2}(1-\xi)\\ \frac{1}{2}(1-\xi)\end{bmatrix}\begin{bmatrix}-\frac{1}{l^e} & \frac{1}{l^e}\end{bmatrix}d\xi = \begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\rightarrow K = \begin{bmatrix}3 & -3 & 0 & 0\\ -3 & 6 & -3 & 0\\ 0 & -3 & 6 & -3\\ 0 & 0 & -3 & 3\end{bmatrix} \quad M = \begin{bmatrix}1/9 & 1/18 & 0 & 0\\ 1/18 & 2/9 & 1/18 & 0\\ 0 & 1/18 & 2/9 & \frac{1}{18}\\ 0 & 0 & 1/18 & 1/9\end{bmatrix}$$

$$C = \begin{bmatrix}-\frac{1}{2} & \frac{1}{2} & 0 & 0\\ -\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} & \frac{1}{2} & 0\\ 0 & -\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} & \frac{1}{2}\\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2}\end{bmatrix} \qquad f^e = \int_{-1}^{1}\frac{l}{2}\begin{bmatrix}\frac{1}{2}(1-\xi)\\ \frac{1}{2}(1+\xi)\end{bmatrix}d\xi = \begin{bmatrix}1/6\\ 1/6\end{bmatrix}$$

$$\rightarrow f = \begin{bmatrix}1/6\\ 1/3\\ 1/3\\ 1/6\end{bmatrix}$$ → applying and reducing the system with $x_1 = x_4 = 0$

$$\rightarrow \begin{bmatrix}6 + \frac{2}{9\Delta t} & -\frac{5}{2} + \frac{1}{18\Delta t}\\ -\frac{7}{2} + \frac{1}{18\Delta t} & \frac{2}{9\Delta t} + 6\end{bmatrix}\begin{bmatrix}u_2\\ u_3\end{bmatrix}^{n+1} = \begin{bmatrix}1/3\\ 1/3\end{bmatrix} + \frac{1}{\Delta t}\begin{bmatrix}2/9 & 1/18\\ 1/18 & 2/9\end{bmatrix}\begin{bmatrix}u_2\\ u_3\end{bmatrix}^{n}$$

if use an initial step as $u_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

then : $\begin{bmatrix} u_2 \\ u_3 \end{bmatrix}^1 = \begin{bmatrix} 6 + \frac{2}{9\Delta t} & -\frac{5}{2} + \frac{1}{18\Delta t} \\ -\frac{7}{2} + \frac{1}{18\Delta t} & 6 + \frac{2}{9\Delta t} \end{bmatrix}^{-1} \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} = \frac{16\Delta t}{54\Delta t + 5} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2) part 1 :

$\frac{\Delta u_d^{n+1}}{\Delta t} + \mathcal{L}_1(u^n) = 0 \longrightarrow \left(\frac{M}{\Delta t} + C\right) u_d^{n+1} = \frac{M}{\Delta t} u^n$

$\mathcal{L}_1(u) = u \cdot \nabla u$, $u_i^{n+1} = u^n + \Delta u_i^{n+1}$

part 2 :

$\frac{\Delta u_2^{n+1}}{\Delta t} + \mathcal{L}_2(u_i^{n+1}) = f^{n+1} + \frac{M}{\Delta t} u_i^{n+1} \rightarrow \left(\frac{M}{\Delta t} + K\right) u^{n+1} = f^{n+1} + \frac{M}{\Delta t} u_i^{n+1}$

$\mathcal{L}_2(u) = \nabla \cdot K \nabla u$, $\Delta u_2^{n+1} = u^{n+1} - u_i^{n+1}$

for the same initial condition :

$u_i^{n+1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow u^{n+1} = \frac{6}{54t+5} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

6. Fractional step methods

$$M\frac{1}{\delta t}(\hat{U}^{n+1}-U^{n})+K\hat{U}^{n+1}=f-G\tilde{P}^{n+1}$$

$$DM^{-1}GP^{n+1}=\frac{1}{\delta t}D\hat{U}^{n+1}-DM^{-1}G\tilde{P}^{n+1}$$

$$M\frac{1}{\delta t}(U^{n+1}-\hat{U}^{n+1})+\alpha K(U^{n+1}-\hat{U}^{n+1})+G(P^{n+1}-\tilde{P}^{n+1})=0$$

1) $$\begin{cases} M\frac{1}{\delta t}\hat{U}^{n+1}-M\frac{1}{\delta t}U^{n}=f-G\tilde{P}^{n+1} \\ \frac{M}{\delta t}U^{n+1}-\frac{M}{\delta t}\hat{U}^{n+1}+\alpha KU^{n+1}-\alpha K\hat{U}^{n+1}+G(P^{n+1}-\tilde{P}^{n+1})=0 \end{cases}$$

$\downarrow$ if we add the equation

$$\frac{M}{\delta t}(U^{n+1}-U^{n})+\alpha KU^{n+1}+(1-\alpha)K\hat{U}^{n+1}+GP^{n+1}=f$$

if we consider $\alpha=1$ then we see that we can recover the momentum equation. So the optimal value for α is 1.

2) The main source of the error in the Yosida scheme is the constrained continuity equation, which can be stabilized by adding a small perturbation to the continuity equation. It can be one of the following forms:

$$\nabla\cdot u+\varepsilon\frac{\partial\varphi}{\partial t}=0\ ,\quad \varphi|_{t=0}=\varphi_0$$

$$\nabla\cdot u+\varepsilon\varphi=0$$

$$\nabla\cdot u-\varepsilon\Delta\varphi=0\quad \Delta\varphi\cdot n|_{\Gamma}=0$$

7. ALE formulations

1. $\gamma(x,y,z,t) = [2x, ye^{t}, z] \rightarrow \begin{cases} x = Xe^{t} \\ y = Y + e^{t} - 1 \\ z = Z \end{cases}$

$\begin{cases} x_m = X + \alpha t \\ y_m = Y - \beta t \\ z_m = Z \end{cases}$

A) $x_{ALE} = \varphi(X,t) - x(X,t)$

$\rightarrow \gamma_{ALE} = \gamma(\varphi(X,t),t) = [2X + 2\alpha t, (Y - \beta t) e^{t}, Z]$

B) $v_p(X,t) = \frac{\partial x}{\partial t} = [Xe^{t}, e^{t}, 0]$

$v_m(X,t) = \frac{\partial x_{ALE}}{\partial t} = [\alpha, -\beta, 0]$

C) $\frac{d\gamma_{ALE}}{dt} = \frac{\partial \gamma_{ALE}}{\partial t} + (v_p - v_m) \cdot \nabla \gamma(x,t)$

$\rightarrow \frac{\partial \gamma_{ALE}}{\partial t} = [2\alpha, (Y - \beta - \beta t) e^{t}, 0]$

$v_p - v_m = [Xe^{t} - \alpha, e^{t} + \beta, 0]$

$\nabla \gamma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \frac{d\gamma_{ALE}}{dt} = [2(X + \alpha t), e^{t}(Y - \beta t + e^{t}), 0]$

2. Using the Navier-Stokes equation in the ALE formulation is basically using the Eulerian description of the equations and replacing the material velocity with the convective velocity $v_p - v_m$

$\begin{cases} \left.\frac{\partial \rho}{\partial t}\right|_x + v \cdot \nabla \rho = -\rho \nabla \cdot v = 0 \\ \rho \frac{\partial v}{\partial t} + \rho(v \cdot \nabla v) = \nabla \cdot \sigma + \rho b \end{cases}$

which will be replacing $v_p - v_m$ in $\begin{cases} (v_p - v_m) \cdot \nabla \rho \\ \rho((v_p - v_m) \cdot \nabla v) \end{cases}$

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8. Fluid-Structure Interaction

1. The added mass effect is added when the density of the fluid and solid are very close to each other then the partitioning schemes donot work properly. This added mass acts as an additional mass on the degrees of freedom of the interface. The main method used to deal with this problem are the steepest descent, Robin boundary conditions applied to both subdomains and the Aitken relaxation scheme.

2. $\begin{cases} \dfrac{\partial u}{\partial t} - K\dfrac{\partial^2 u}{\partial x^2} = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ $\quad \Omega = [0, 1] \quad f = K = 1$

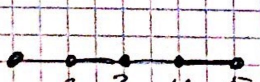
$$\left(\frac{M}{\Delta t} + K\right) u^{n+1} = f + \frac{M}{\Delta t} u^n$$

$$\Omega_1: \begin{cases} \left(\dfrac{M_1}{\Delta t} + K_1\right) u_1^{n+1,k} = f_1 + \dfrac{M_1}{\Delta t} u_1^{n,k-1} & \text{on } \Omega_1 \\ \dfrac{du_{\Gamma_{12}}^{n+1,k}}{dn} = -\dfrac{du_{\Gamma_{21}}^{n+1,k-1}}{dn} & \text{on } \Gamma_{12} \\ u^{n+1,k} = 0 & \text{on } \partial\Omega_1 \end{cases}$$

$$w = \frac{u_{\Gamma_{21}}^{n+1,k-2} - u_{\Gamma_{21}}^{n+1,k-1}}{u_{\Gamma_{21}}^{n+1,k-2} - u_{\Gamma_{21}}^{n+1,k-1} + u_{\Gamma_{12}}^{n+1,k} - u_{\Gamma_{12}}^{n+1,k-1}}$$

$$\Omega_2: \begin{cases} \left(\dfrac{M_2}{\Delta t} + K_2\right) u_2^{n+1,k} = f_2 + \dfrac{M_2}{\Delta t} u_2^{n,k-1} & \text{on } \Omega_2 \\ u_{\Gamma_{21}}^{n+1,k} = w\, u_{\Gamma_{12}}^{n+1,k} + (1-w)\, u_{\Gamma_{21}}^{n+1,k-1} & \text{on } \Gamma_{21} \\ u_2^{n+1,k} = 0 & \text{on } \partial\Omega_2 \end{cases}$$

Because the relaxation parameter w needs two iteration we need to start the iterations with another method and from the second iteration use the Aitkens scheme.

3. 

$h = 1/4$

for each element we'll have:

$$K^e = \frac{K}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}, \quad M^e = h\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/12 & 1/24 \\ 1/24 & 1/12 \end{bmatrix}$$

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After assembling:

$$K = \begin{bmatrix} 4 & -4 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix} \qquad M = \begin{bmatrix} 1/12 & 1/24 & 0 & 0 & 0 \\ 1/24 & 1/6 & 1/24 & 0 & 0 \\ 0 & 1/24 & 1/6 & 1/24 & 0 \\ 0 & 0 & 1/24 & 1/6 & 1/24 \\ 0 & 0 & 0 & 1/24 & 1/12 \end{bmatrix}$$

Building the Lagrange multipliers for the Dirichlet boundary condition

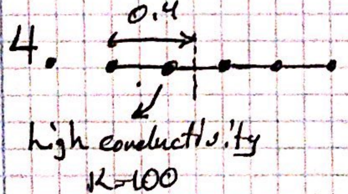
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{L} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_5 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{b}$$

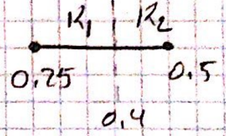
$$A = \frac{M}{\Delta t} + K \quad , \quad F = f + \frac{M}{\Delta t} u^n$$

$$\rightarrow \begin{bmatrix} A & L^T \\ L & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ b \end{bmatrix}$$

The condition number of the A matrix using matlab is obtained as $K_A = 36.87$

4.





$$\rightarrow K_{22}^{(2)} = K_{11}^{(2)} = K_1 \int_{0.25}^{0.4} \left(\frac{-x}{1/4}\right)\left(\frac{-x}{1/4}\right) dx + K_2 \int_{0.4}^{0.5} \left(-\frac{x}{1/4}\right)\left(-\frac{x}{1/4}\right) dx = 242.4$$

$$K_{21}^{(2)} = K_{12}^{(2)} = -242.4$$

So Assembling we will have:

$$K = \begin{bmatrix} 400 & -400 & 0 & 0 & 0 \\ -400 & 642.4 & -242.4 & 0 & 0 \\ 0 & -242.4 & 246.4 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}$$

the M matrix will be the same as point 3 and the system to be solved will be:

$$\left(\frac{M}{\Delta t} + K\right) u^{n+1} = f + \frac{M}{\Delta t} u^n$$

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