Finite deformation nonlinear elasticity

Kirchhoff Saint-Venant material model

Isotropic linear elasticity can be derived from balance of linear momentum, the linearized strain displacement relation $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$, and the stored elastic energy function

$$W(\boldsymbol{\varepsilon}) = \frac{\lambda}{2}(\operatorname{tr}\boldsymbol{\varepsilon})^2 + \mu \operatorname{tr}(\boldsymbol{\varepsilon}^2) = \frac{\lambda}{2}(\varepsilon_{ii})^2 + \mu\, \varepsilon_{jk}\varepsilon_{jk}.$$

(1) **Check that, the stress tensor obtained from $\boldsymbol{\sigma} = \partial W/\partial \boldsymbol{\varepsilon}$ agrees with the usual linear elasticity expression.**

The usual linear elasticity expression in Einstein notation has the form

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}. \tag{1}$$

So rewriting the derivative of the elastic energy equation with different indices yields

$$\begin{aligned}
\sigma_{ij} &= \partial W/\partial \varepsilon_{ij} \\
&= \frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{\lambda}{2}(\varepsilon_{kk})^2 + \mu(\varepsilon_{kl}\,\varepsilon_{kl})\right) \\
&= \frac{2}{2}\lambda\varepsilon_{kk}\frac{\partial \varepsilon_{kk}}{\partial \varepsilon_{ij}} + \mu\left(\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{ij}}\varepsilon_{kl} + \varepsilon_{kl}\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{ij}}\right) \\
&= \lambda\varepsilon_{kk}\delta_{ki}\delta_{kj} + \mu(\varepsilon_{kl} + \varepsilon_{kl})\delta_{ik}\delta_{lj} \\
&= \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}
\end{aligned}$$

which is equal to equation (1) and therefore agrees with the usual linear ealsticity expression.

Since the linearization of the Green-Lagrange strain tensor $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{Id})$ is the small strain tensor $\boldsymbol{\varepsilon}$, it is natural to extend isotropic elasticity to nonlinear elasticity as

$$W(\boldsymbol{E}) = \frac{\lambda}{2}(\operatorname{tr}\boldsymbol{E})^2 + \mu \operatorname{tr}(\boldsymbol{E}^2). \tag{2}$$

This hyperelastic model is called Kirchhoff Saint-Venant material model.

(2) According to the definition we gave in class about isotropy in nonlinear elasticity, is this model isotropic?

According to the definition given in the lecture, a material is isotropic, if any rotation is a material symmetry. To check if the Kirchhoff Saint-Venant material model is isotropic, the Representation theorem can be used to proof objectivity. It states, that any isotropic strain energy function can be written in terms of the principal invariants of $\boldsymbol{C}$.

$$W(\boldsymbol{C}) = W(I_1(\boldsymbol{C}), I_2(\boldsymbol{C}), I_3(\boldsymbol{C}))$$

$$I_1(\boldsymbol{C}) = \operatorname{trace} \boldsymbol{C}$$
$$I_2(\boldsymbol{C}) = \frac{1}{2}[(\operatorname{trace} \boldsymbol{C})^2 - \operatorname{trace} \boldsymbol{C}^2]$$
$$I_3(\boldsymbol{C}) = \det \boldsymbol{C} = J^2$$

When inserting $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{Id})$ into equation (2)

$$W(\boldsymbol{C}) = \frac{\lambda}{8}(\operatorname{tr}(\boldsymbol{C} - \boldsymbol{Id}))^2 + \frac{\mu}{4}\operatorname{tr}((\boldsymbol{C} - \boldsymbol{Id})^2)$$

we can see that the function $W(\boldsymbol{C})$ still just depends on the principal invariants of $\boldsymbol{C}$ and thus, is objective.

(3) Derive the second Piola-Kirchhoff stress $\boldsymbol{S}$.

From power-conjugacy relations we know that

$$\boldsymbol{S} = 2\frac{\partial W}{\partial \boldsymbol{C}} = \frac{\partial W}{\partial \boldsymbol{E}}. \tag{3}$$

Similar to the first task we can write the extended elastic energy function as

$$W(\boldsymbol{E}) = \frac{\lambda}{2}(E_{ii})^2 + \mu\, E_{jk}E_{jk}$$

and obtain the second Piola-Kirchhoff stress using the equation (3).

$$\begin{aligned}
\boldsymbol{S} &= \frac{\partial W}{\partial E_{ij}} \\
&= \frac{\partial}{\partial E_{ij}}\left(\frac{\lambda}{2}(E_{kk})^2 + \mu(E_{kl}\,E_{kl})\right) \\
&= \frac{2}{2}\lambda E_{kk}\frac{\partial E_{kk}}{\partial E_{ij}} + \mu\left(\frac{\partial E_{kl}}{\partial E_{ij}}E_{kl} + E_{kl}\frac{\partial E_{kl}}{\partial E_{ij}}\right) \\
&= \lambda E_{kk}\delta_{ki}\delta_{kj} + \mu(E_{kl} + E_{kl})\delta_{ik}\delta_{lj} \\
&= \lambda E_{kk}\delta_{ij} + 2\mu E_{ij}
\end{aligned}$$

Rewriting the result in general form and inserting $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{Id})$ yields

$$\boldsymbol{S} = \lambda(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E} = \lambda(\operatorname{tr}(\boldsymbol{C} - \boldsymbol{Id}))\boldsymbol{I} + 2\mu(\boldsymbol{C} - \boldsymbol{Id}). \tag{4}$$

(4) **For a uniform deformation of a rod aligned with the X axis ($x = \Lambda X$, $y = Y$, $z = Z$, where $\Lambda > 0$ is the stretch ratio along the X direction) derive the relation between the nominal normal stress P (the xX component of the first Piola-Kirchhoff stress) and the stretch ratio Λ, $P(\Lambda)$, and plot it.**

The deformation mapping is given as

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}) = \{\Lambda X, Y, Z\}.$$

So the deformation gradient $\boldsymbol{F}$ is calculated as

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In order to get the Piola-Kirchhoff stress tensor, the Right Cauchy Stress tensor and the Green-Lagrange strain have to be computed as follows:

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \Lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{Id}) = \frac{1}{2}\left(\begin{bmatrix} \Lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} (\Lambda^2 - 1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the relation $\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P}$ and equation (4) we can recover the xX component of the first Piola-Kirchhoff stress:

$$\begin{aligned} P_{xX} &= F_{xX}S_{xX} = F_{xX}(\lambda E_{xX} + 2\mu E_{xX}) \\ P(\Lambda) &= \Lambda(\lambda\frac{1}{2}(\Lambda^2 - 1) + 2\mu\frac{1}{2}(\Lambda^2 - 1)) \\ &= \frac{\Lambda}{2}(\Lambda^2 - 1)(\lambda + 2\mu) \end{aligned}$$

The plot in Figure 1 shows the xX component of the first Piola-Kirchhoff stress in dependence of the stretch ratio Λ for $\lambda = 2$ and $\mu = 3$.

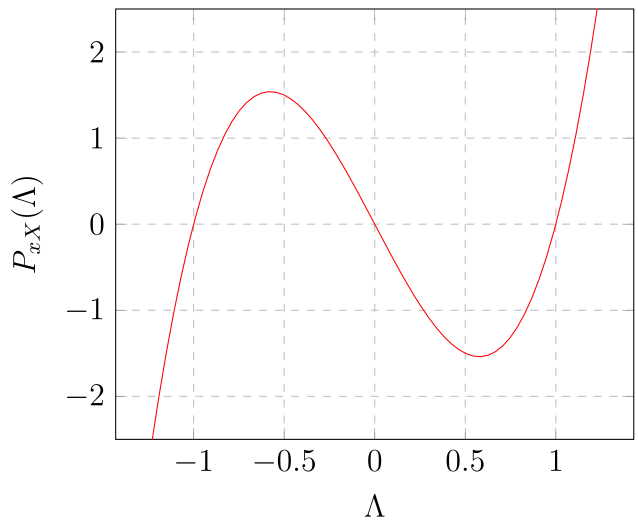

Figure 1: Caption

(5) Is the relation $P(\Lambda)$ monotonic? If not, derive the critical stretch Λ_{crit} at which the model fails with zero stiffness. Does this critical stretch depend on the elastic constants? Show that the material does not satisfy the growth conditions

$$W(\boldsymbol{E}) \to +\infty \quad \text{when} \quad J \to 0^+.$$

Discuss your answers.

A function is monotonic when it is always non-decreasing or non-increasing. In other words, the function can not have any local minima or maxima, so the derivative can not change its sign. By looking at Figure 1 it is obvious that the relation $P(\Lambda)$ is not monotonic. To find the critical stretch Λ_{crit} we set the derivative of $P(\Lambda)$ equal to zero:

$$\frac{dP_{xX}(\Lambda)}{d\Lambda} = \frac{1}{2}(3\Lambda^2 - 1)(\lambda + 2\mu) = 0$$

$$\Rightarrow \Lambda_{crit} = \pm\frac{\sqrt{3}}{3}$$

Because the negative value has no physical meaning ($\Lambda > 0$), $\Lambda_{crit} = \frac{\sqrt{3}}{3}$ or $\lambda = \mu = 0$ are the only possible solutions. Since the later is a trivial solution, we can say, that the critical stretch does not depend on the elastic constants. In order to check if the material satisfies the growth conditions the Jacobian J and the stored elastic energy $W(\boldsymbol{E})$ have to be computed.

$$J = \det(\boldsymbol{F}) = \det\left(\begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \Lambda$$

$$W(\boldsymbol{E}) = \frac{\lambda}{2}(\operatorname{tr}\boldsymbol{E})^2 + \mu\operatorname{tr}(\boldsymbol{E}^2)$$

$$(\operatorname{tr}\boldsymbol{E})^2 = (\frac{1}{2}(\Lambda^2-1))^2 = \frac{1}{4}(\Lambda^2-1)^2$$

$$\operatorname{tr}(\boldsymbol{E}^2) = \operatorname{tr}\left(\frac{1}{4}\begin{bmatrix}(\Lambda^2-1)^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right) = \frac{1}{4}(\Lambda^2-1)^2$$

$$\begin{aligned} W(\boldsymbol{E}) &= \frac{\lambda}{2}(\frac{1}{4}(\Lambda^2-1)^2) + \mu\frac{1}{4}(\Lambda^2-1)^2 \\ &= \frac{\lambda}{8}(\Lambda^2-1)^2 + \frac{\mu}{4}(\Lambda^2-1)^2 \\ &= (\Lambda^2-1)^2\,(\frac{\lambda}{8}+\frac{\mu}{4}) \end{aligned}$$

Since the Jacobian J is equal to the stretch parameter Λ, it can easily be substituted into the derived energy equation.

$$W(\boldsymbol{E}) = (J^2-1)^2\,(\frac{\lambda}{8}+\frac{\mu}{4})$$

Checking the growth condition yields

$$\lim_{J\to 0^+} W(\boldsymbol{E}) = ((0^+)^2-1)^2\,(\frac{\lambda}{8}+\frac{\mu}{4}) = (\frac{\lambda}{8}+\frac{\mu}{4}).$$

So for $J \to 0^+$, which can be interpreted as a shrinking in volume, the energy $W(\boldsymbol{E})$ approaches a finite value depending on the elastic constants and therefore the Kirchhoff Saint-Venant material model does not fulfill the growth conditions. This fact shows, that the model fails to represent very large compressive deformations ($\Lambda \to 0^+$).

(6) Consider now the modified Kirchhoff Saint-Venant material model:

$$W(\boldsymbol{E}) = \frac{\lambda}{2}(\ln J)^2 + \mu\operatorname{tr}(\boldsymbol{E}^2).$$

Does this model circumvent the drawbacks of the previous model?

Again, we check the growth condition for the modified Kirchhoff Saint-Venant material model by rewriting the whole energy equation in terms of J and checking the limit state $J \to 0^+$.

$$\begin{aligned} W(\boldsymbol{E}) &= \frac{\lambda}{2}(\ln J)^2 + \mu\operatorname{tr}(\boldsymbol{E}^2) \\ &= \frac{\lambda}{2}(\ln J)^2 + \frac{\mu}{4}(J^2-1)^2 \\ \lim_{J\to 0^+} W(\boldsymbol{E}) &= \frac{\lambda}{2}(\ln 0^+)^2 + \frac{\mu}{4}((0^+)^2-1)^2 = \infty \end{aligned}$$

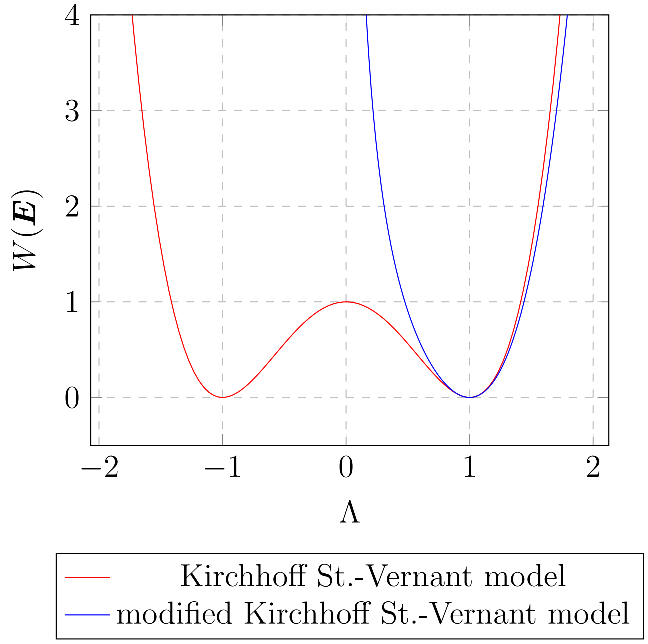

Figure 2: Caption

As shown the modified version of the Kirchhoff Saint-Venant material model now satisfies the growth condition and is also able to represent large compressive forces respectively a large shrinking in volume. To further compare both models the energy $W(\boldsymbol{E})$ with respect to the stretch ratio Λ (respectively the Jacobian J) is plotted in Figure 2, both for the original and the modified model. In comparison to the original model the energy of the modified model tends to go to infinity when Λ approaches zero and so overcomes the drawbacks of the standard Kirchhoff Saint-Venant material model.