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Computational Solid Mechanics
Màster en Mètodes Numèrics

HW1C

Consider a cylindrical solid referred to an orthonormal Cartesian reference frame $\{x^1, x^2, x^3\}$, whose axis aligned with the x^3 direction. Its normal cross section occupies a region Ω in the $\{x^1, x^2\}$ plane of boundary $\partial\Omega$. An anti-plane shear deformation of the solid can be defined as one for which the deformation mapping is of the form: $\varphi_1 = x^1$, $\varphi_2 = x^2$, $\varphi_3 = x^3 + \omega(x^1, x^2)$.

Spatial and material reference frames are taken to coincide, and the function ω is defined over Ω.

(1) a). Compute the deformation gradient field F, right Cauchy-Green deformation tensor C, and the Jacobian J of the deformation field in terms of ω

Recall that F is:

$$\underline{\underline{F}} = \begin{bmatrix} \frac{\partial\varphi^1}{\partial x^1} & \frac{\partial\varphi^1}{\partial x^2} & \frac{\partial\varphi^1}{\partial x^3} \\ \frac{\partial\varphi^2}{\partial x^1} & \frac{\partial\varphi^2}{\partial x^2} & \frac{\partial\varphi^2}{\partial x^3} \\ \frac{\partial\varphi^3}{\partial x^1} & \frac{\partial\varphi^3}{\partial x^2} & \frac{\partial\varphi^3}{\partial x^3} \end{bmatrix}$$

then, the deformation gradient tensor is:

$$\underline{\underline{F}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial\omega}{\partial x^1} & \frac{\partial\omega}{\partial x^2} & 1 \end{bmatrix}$$

note that, the unit values of the diagonal depict rotation whereas off-diagonal terms show shear deformation.

For C, Right-Cauchy-Green tensor, we know that, by definition:

$C = F^T F$; therefore:

$$\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}} = \begin{bmatrix} 1 & 0 & \frac{\partial\omega}{\partial x^1} \\ 0 & 1 & \frac{\partial\omega}{\partial x^2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial\omega}{\partial x^1} & \frac{\partial\omega}{\partial x^2} & 1 \end{bmatrix} = \begin{bmatrix} 1+\left(\frac{\partial\omega}{\partial x^1}\right)^2 & \frac{\partial\omega}{\partial x^1}\frac{\partial\omega}{\partial x^2} & \frac{\partial\omega}{\partial x^1} \\ \frac{\partial\omega}{\partial x^2}\cdot\frac{\partial\omega}{\partial x^1} & 1+\left(\frac{\partial\omega}{\partial x^2}\right)^2 & \frac{\partial\omega}{\partial x^2} \\ \frac{\partial\omega}{\partial x^1} & \frac{\partial\omega}{\partial x^2} & 1 \end{bmatrix}$$

It is symmetric as it is refered from its own definition

On the last point, J, the jacobian of the deformation field.

$\Rightarrow$ the Jacobian is defined as: $J := \det(F)$

$$J = \det(\underline{\underline{F}}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial w}{\partial X^1} & \frac{\partial w}{\partial X^2} & 1 \end{vmatrix} = 1$$

3) By definition, $J = 1$ corresponds to an isochoric deformation. This means that the deformation is produced with no change in the volume. Therefore, spatial volume dv is equivalent to dV, $dv = dV$

c) Are the local impenetrability conditions satisfied?

Since $J > 0$, there exists an inverse mapping deformation such that a spatial point is related to a material point and this transformation is unique.

(2) Consider the unit vectors:

$$\underline{A} = \frac{w_{,1} E_1 + w_{,2} E_2}{\sqrt{w_{,1}^2 + w_{,2}^2}} \qquad \underline{B} = \frac{-w_{,2} E_1 + w_{,1} E_2}{\sqrt{w_{,1}^2 + w_{,2}^2}}$$

where $\{E_I\}$, $I = 1, 2, 3$ are the orthonormal ~~vectors~~ material basis vectors.

a) How are A and B related to the level contours of $w(X_1, X_2)$?

$\Rightarrow$ Assuming that $w(X^1, X^2)$ is a contour level s.t. is constant by definition, $w(X^1, X^2) = ct.$, it is possible to define its gradient as: $\nabla \underline{w} = w_{,1} E_1 + w_{,2} E_2$, being its module: $\| \nabla \underline{w} \| = \sqrt{(w_{,1} E_1)^2 + (w_{,2} E_2)^2}$

Therefore the vector $\underline{A}$, defined previously is nothing but:

$$\underline{A} = \frac{\nabla \underline{w}}{\| \nabla \underline{w} \|}$$ $\Rightarrow$ unit normal vector over the contour level. $w(X^1, X^2) = ct.$

Since $\underline{A} \cdot \underline{B} = 0$ $\Rightarrow$ $\underline{A}$ and $\underline{B}$ are perpendicular to each other.

So $\underline{B}$ is a unit tangent vector of $w(X^1, X^2) = ct.$

b) compute (in terms of w) the change in length (measured by the corresponding stretch ratios) of $\underline{A}$ and $\underline{B}$ as well as the change in the angle subtended by $\underline{A}$ and $\underline{B}$

As seen in class, the stretch ratio is defined as:

$$\frac{d\underline{a}}{d\underline{A}} = \sqrt{\underline{N_A} \cdot \underline{\underline{C}} \cdot \underline{N_A}^T}$$

$\underline{\underline{C}}$: Right Green-Cauchy tensor.

$\underline{N_A}$: material unit vector of $\underline{A}$

So, the change in length of vector $\underline{A}$ is computed as:

Taking:

$$\underline{N_A} = \underline{A} \quad \text{and} \quad \frac{d\underline{a}}{d\underline{A}} = \sqrt{\underline{N_A} \cdot \underline{\underline{C}} \cdot \underline{N_A}^T} =$$

$$= \frac{1}{\sqrt{\left(\frac{dw}{dx^1}\right)^2 + \left(\frac{dw}{dx^2}\right)^2}} \cdot \sqrt{\left[\frac{dw}{dx^1}, \frac{dw}{dx^2}, 0\right] \begin{bmatrix} 1+\left(\frac{dw}{dx^1}\right)^2 & \frac{dw}{dx^1}\frac{dw}{dx^2} & \frac{dw}{dx^1} \\ \frac{dw}{dx^2}\frac{dw}{dx^1} & 1+\left(\frac{dw}{dx^2}\right)^2 & \frac{dw}{dx^2} \\ \frac{dw}{dx^1} & \frac{dw}{dx^2} & 1 \end{bmatrix} \begin{bmatrix} \frac{dw}{dx^1} \\ \frac{dw}{dx^2} \\ 0 \end{bmatrix}} =$$

$$\Rightarrow \frac{d\underline{a}}{d\underline{A}} = \sqrt{\left(\frac{dw}{dx^1}\right)^2 + \left(\frac{dw}{dx^2}\right)^2 + 1}$$

then, if we proceed similarly, $\underline{N_B} = \underline{B}$

$$\frac{d\underline{b}}{d\underline{B}} = \sqrt{\underline{N_B} \cdot \underline{\underline{C}} \cdot \underline{N_B}^T} = \frac{1}{\sqrt{\left(\frac{dw}{dx^1}\right)^2 + \left(\frac{dw}{dx^2}\right)^2}} \sqrt{\left[-\frac{dw}{dx^2}, \frac{dw}{dx^1}, 0\right] \begin{bmatrix} 1+\left(\frac{dw}{dx^1}\right)^2 & \frac{dw}{dx^1}\frac{dw}{dx^2} & \frac{dw}{dx^1} \\ \frac{dw}{dx^2}\frac{dw}{dx^1} & 1+\left(\frac{dw}{dx^2}\right)^2 & \frac{dw}{dx^2} \\ \frac{dw}{dx^1} & \frac{dw}{dx^2} & 1 \end{bmatrix} \begin{bmatrix} -\frac{dw}{dx^2} \\ \frac{dw}{dx^1} \\ 0 \end{bmatrix}}$$

$$\Rightarrow \frac{d\underline{b}}{d\underline{B}} = 1$$

For the calculation of the angle, we define:

$$\cos(\underline{A},\underline{B}) = \frac{\underline{N_B} \cdot \underline{\underline{C}} \cdot \underline{N_A}^T}{\left(\frac{d\underline{a}}{d\underline{A}}\right)\left(\frac{d\underline{b}}{d\underline{B}}\right)}$$

if we compute separately numerator and denominator, we have the following:

$$\underline{N_B} \cdot \underline{\underline{C}} \cdot \underline{N_A}^T = \left[-\frac{dw}{dx^2}, \frac{dw}{dx^1}, 0\right] \begin{bmatrix} 1+\left(\frac{dw}{dx^1}\right)^2 & \frac{dw}{dx^1}\frac{dw}{dx^2} & \frac{dw}{dx^1} \\ \frac{dw}{dx^2}\frac{dw}{dx^1} & 1+\left(\frac{dw}{dx^2}\right)^2 & \frac{dw}{dx^2} \\ \frac{dw}{dx^1} & \frac{dw}{dx^2} & 1 \end{bmatrix} \begin{bmatrix} \frac{dw}{dx^1} \\ \frac{dw}{dx^2} \\ 0 \end{bmatrix} = 0$$

For the denominator:

$$\left(\frac{d\underline{a}}{d\underline{A}}\right)\left(\frac{d\underline{b}}{d\underline{B}}\right) = \left[\sqrt{\left(\frac{dw}{dx^1}\right)^2+\left(\frac{dw}{dx^2}\right)^2+1}\right]\cdot 1 = \sqrt{\left(\frac{dw}{dx^1}\right)^2+\left(\frac{dw}{dx^2}\right)^2+1}$$

The calculation for angle reads:

$$\cos(\underline{A},\underline{B}) = \frac{0}{\sqrt{\left(\frac{dw}{dx^1}\right)^2+\left(\frac{dw}{dx^2}\right)^2+1}} = 0 \Rightarrow \vartheta = 90°$$

c) Interpret the results.

1. $\underline{A}$ and $\underline{B}$ vectors are orthogonal in spatial and material configurations.
2. $\underline{A}$ vector extends itself in the spatial configuration, while $\underline{B}$ vector remains the same in both spatial and material configurations.
3. In terms of length-change, tangent vector remain constant whereas normal vector changes. This is in close relation to antiplane shear which is achieved when the displacements in the body (solid) are 0 in the plane of interest, but non-zero in the direction perpendicular to the plane.

(3) Using the Piola transformation, compute (in terms of w) the change in area of, and in the normal to, an infinitesimal material area contained in the $\{X^1, X^2\}$ plane.

By definition we know that:

$$d\underline{s} = J\underline{\underline{F}}^{-T} d\underline{S}$$

$d\underline{s}$ → spatial differential area

J → Jacobian

$\underline{\underline{F}}^{-T}$ → inverse of the transpose of deformation gradient tensor.

$d\underline{S}$ → material differential area.

Assuming isochoric deformation where $J=1$ and material unit vector for $d\underline{S}$ by requirements of the statement. $\Rightarrow \underline{N}_{dS} = [0, 0, 1]$

the expression for $d\underline{s}$ yields to:

$$d\underline{s} = 1 \cdot \begin{bmatrix} 1 & 0 & -\frac{dw}{dx^1} \\ 0 & 1 & -\frac{dw}{dx^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dS = \begin{bmatrix} -\frac{dw}{dx^1} \\ -\frac{dw}{dx^2} \\ 1 \end{bmatrix} dS$$

(5) Derive an integral expression for the deformed area of the domain Ω

From the expression used before we have:

$d\underline{s} = J\underline{\underline{F}}^{-T} d\underline{S}$, if we integrate over the domain Ω $\Rightarrow \int_{\Omega^0} d\underline{s} = \int_{\Omega^0} \| J\underline{\underline{F}}^{-T} d\underline{S} \| d\Omega \Rightarrow$

$$\Rightarrow \int_{\Omega^0} d\underline{s} = \int_{\Omega^0} \left\| 1 \cdot \begin{bmatrix} 1 & 0 & -\frac{dw}{dx^1} \\ 0 & 1 & -\frac{dw}{dx^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\| dS = \int_{\Omega^0} \sqrt{\left(\frac{dw}{dx^1}\right)^2 + \left(\frac{dw}{dx^2}\right)^2 + 1} \, dS$$

⑥ Let the boundary $\partial\Omega$ of Ω be defined parametrically by the equations:

$$x^1 = x^1(s) \qquad x^2 = x^2(s)$$

where $0 \leq s \leq L$ is the arc-length measured along $\partial\Omega$. Note that

$\dfrac{E_1\, x^1(s)}{ds} + \dfrac{E_2\, x^2(s)}{\partial s}$ is the unit vector tangent to $\partial\Omega$. Derive an integral

expression for the perimeter of the deformed boundary $\varphi(\partial\Omega)$

If we recall to continuum mechanics, the definition of the arc-length of a boundary can be stated as:

$$\int_{d\Omega} dl = \int_{d\Omega_0} \lambda \, dL = \int_{d\Omega_0} \sqrt{1 + 2\underline{T}\cdot\underline{\underline{E}}\cdot\underline{T}}\; dL$$

$\underline{T} \Rightarrow$ unit vector along material direction $\Rightarrow \underline{T} = \begin{bmatrix} \dfrac{x^1(s)}{ds} & \dfrac{x^2(s)}{ds} & 0 \end{bmatrix}$

$\underline{\underline{E}} \Rightarrow$ Green-Lagrange strain tensor. $\Rightarrow \underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{G}})$

↳ where $\underline{\underline{G}}$ is a metric tensor $\underline{\underline{G}} = \underline{\underline{I}}$

$$\underline{\underline{E}} = \frac{1}{2}\left[\begin{bmatrix} 1 + \left(\dfrac{dw}{dx^1}\right)^2 & \dfrac{dw}{dx^1}\dfrac{dw}{dx^2} & \dfrac{dw}{dx^1} \\ \dfrac{dw}{\partial x^2}\dfrac{\partial w}{\partial x^1} & 1 + \left(\dfrac{\partial w}{\partial x^2}\right)^2 & \dfrac{\partial w}{\partial x^2} \\ \dfrac{\partial w}{\partial x^1} & \dfrac{\partial w}{\partial x^2} & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] =$$

$$= \frac{1}{2}\begin{bmatrix} \left(\dfrac{\partial w}{dx^1}\right)^2 & \dfrac{dw}{\partial x^1}\dfrac{\partial w}{dx^2} & \dfrac{\partial w}{dx^1} \\ \dfrac{\partial w}{\partial x^2}\dfrac{\partial w}{\partial x^1} & \left(\dfrac{\partial w}{\partial x^2}\right)^2 & \dfrac{\partial w}{\partial x^2} \\ \dfrac{\partial w}{\partial x^1} & \dfrac{\partial w}{\partial x^2} & 0 \end{bmatrix}$$

Therefore, if we compute terms together,

$$\sqrt{1+2\underline{T}\cdot\underline{\underline{E}}\cdot\underline{T}} = \sqrt{1+2\begin{bmatrix}\frac{x^1(s)}{\partial s} & \frac{x^2(s)}{\partial s} & 0\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}\left(\frac{\partial w}{\partial x^1}\right)^2 & \frac{\partial w}{\partial x^1}\frac{\partial w}{\partial x^2} & \frac{\partial w}{\partial x^1}\\ \frac{\partial w}{\partial x^2}\frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & \frac{\partial w}{\partial x^2}\\ \frac{\partial w}{\partial x^1} & \frac{\partial w}{\partial x^2} & 0\end{bmatrix}\begin{bmatrix}\frac{x^1(s)}{ds}\\ \frac{x^2(s)}{ds}\\ 0\end{bmatrix}} =$$

$$= \sqrt{\left(\frac{x^1(s)}{ds}\frac{\partial w}{\partial x^1}+\frac{x^2(s)}{ds}\frac{\partial w}{\partial x^2}\right)^2+1}$$

~~the~~ thus, the above expression for the perimeter of the deformed boundary is:

$$\int_{d\Omega} dl = \int_{d\Omega^0}\sqrt{1+2\underline{T}\cdot\underline{\underline{E}}\cdot\underline{T}}\,ds = \int_{d\Omega_0}\sqrt{\left(\frac{x^1(s)}{ds}\frac{\partial w}{\partial x^1}+\frac{x^2(s)}{ds}\frac{\partial w}{\partial x^2}\right)^2+1}\,ds$$